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COMPLEX ROOTS OF A TRANSCENDENTAL EQUATION

BY H. GOLDENBERG.

A joint paper by Tranter and the present author [1] concerning the heat flow in an infinite medium heated by a sphere has necessitated a proof that no complex zeros of the equation

$$\coth z = (c/z) - b \quad \left.\begin{matrix} c<1 \\ b>0 \end{matrix}\right\} \qquad \ldots\ldots(1)$$

possess a positive real part. This proof has not been included in the above paper and does not appear to have been given elsewhere. Further it is not readily obtainable from the existing complex variable theorems concerning the number of zeros of a function within a contour.[2] The zeros of (1), which is a particular case of

$$\exp z = \frac{pz+q}{rz+s} \qquad \ldots\ldots(2)$$

are considered in a manner somewhat different from that of Hayes,[3] who encountered a consideration of the nature of the zeros of another particular case of (2) in connection with the stability of solutions of certain types of difference-differential equations.

The author's consideration of the nature of the complex zeros of (1) is somewhat different from that of previous authors, and, apart from giving rise to a method for the determination of the exact location at which such zeros exist, includes what appears to be a new method for the numerical solution of two simultaneous transcendental equations. Consequently it is felt worthwhile to present an account of the method used in the consideration of the nature of the zeros of (1).

With $z = x + iy$, the separation of (1) into real and imaginary parts gives

$$\frac{\sinh x \cosh x}{\sinh^2 x + \sin^2 y} = \frac{cx}{x^2+y^2} - b \qquad \ldots\ldots(3)$$

$$\frac{\sin y \cos y}{\sinh^2 x + \sin^2 y} = \frac{cy}{x^2+y^2} \qquad \ldots\ldots(4)$$

and corresponding to each solution $z = x + iy$ of (1), there is a solution $\bar{z} = x - iy$ as (1) contains only real coefficients.

An alternative form of (1) is

$$e^{2z}=\frac{(b-1)z-c}{(b+1)z-c}. \quad \ldots\ldots(5)$$

Equating the modulus of each side of (5), we have

$$e^{2x}=\left\{\frac{[(b-1)x-c]^2+y^2(b-1)^2}{[(b+1)x-c]^2+y^2(b+1)^2}\right\}^{1/2} \quad \ldots\ldots(6)$$

and it is apparent that solutions of (6) with $x>0$ cannot exist for

$$[(b+1)x-c]^2+y^2(b+1)^2>[(b-1)x-c]^2+y^2(b-1)^2$$

With $b>0$ this inequality can be written

$$x[x-(c/b)]+y^2>0$$

and therefore if $c<0$, no solutions exist with $x>0$, and if $c>0$ no solutions exist with $x>c/b$. There is thus an upper bound to the real part of the solutions of (1), and it remains to show that when $0\leqslant c<1$, no solution $z=x+iy$ of (1) exists with $x>0$. Before proceeding with this proof it will be advantageous to extract some further useful information from (5).

The separation of (5) into real and imaginary parts gives

$$e^{2x}\cos 2y=\frac{b-1}{b+1}+O\left(\frac{1}{y^2}\right). \quad \ldots\ldots(7)$$

$$e^{2x}\sin 2y=O\,(1/y). \quad \ldots\ldots(8)$$

From (7) and (8) it may be concluded that should there exist an infinite number of solutions of (1), then for $b>1$ and for sufficiently large values of $|\,y\,|$, the solutions tend more and more closely to be situated at

$$x=-\tfrac{1}{2}\log_e\left(\frac{b+1}{b-1}\right), \quad |\,y\,|=n\pi \quad \ldots\ldots(9)$$

while for $b<1$ and for sufficiently large values of $|\,y\,|$, the solutions tend more and more closely to be situated at

$$x=-\tfrac{1}{2}\log_e\left(\frac{1+b}{1-b}\right), \quad |\,y\,|=(n+\tfrac{1}{2})\pi \quad \ldots\ldots(10)$$

where n is a sufficiently large positive integer.

Proceeding with the proof that when $0\leqslant c<1$, no solution $z=x+iy$ of (1) exists with $x>0$, it may be seen that equations (3) and (4) are equivalent to

$$\frac{\sinh 2x}{\sin 2y}=\frac{x-(b/c)(x^2+y^2)}{y} \quad \ldots\ldots(11)$$

and

$$\frac{\sin 2y}{\cosh 2x-\cos 2y}=\frac{cy}{x^2+y^2}. \quad \ldots\ldots(12)$$

The relation between x and y given by either of these equations can be shown graphically as the locus of the intersections of the separate curves given by equating each side of that equation to all possible real constants. For example, the relation (11) is given as the locus of the intersection of the curves

$$\frac{\sinh 2x}{\sin 2y}=k, \quad \ldots\ldots(13)$$

$$x-(b/c)(x^2+y^2)=ky. \quad \ldots\ldots(14)$$

The curves represented by (13) are of period π in y and when

$0<y<\pi/2$,	$x>0$ for $k>0$,	$x<0$ for $k<0$,
$y=\pi/2$,	$x=0$,	
$\pi/2<y<\pi$	$x<0$ for $k>0$,	$x>0$ for $k<0$.

Within the range $0<y<\pi$, the curves represented by (13) are of the forms shown in Fig. 1 below.

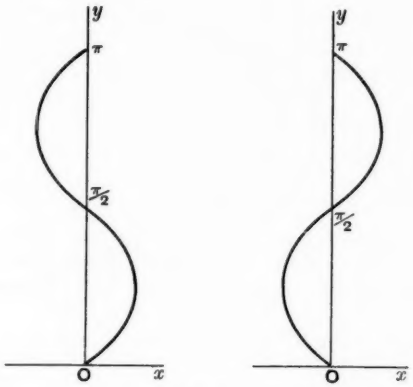

FIG. 1.

(*a*) $k>0$ (*b*) $k<0$

Equation (14) can be written

$$\left(x-\frac{c}{2b}\right)^2+\left(y+\frac{kc}{2b}\right)^2=\left(\frac{c}{2b}\right)^2(1+k^2)$$

so that it represents a circle passing through the origin with centre at $(c/2b, -kc/2b)$. Further the gradients of the tangents at the origin to this circle and the curve represented by (13) are each equal to $1/k$. Hence when $y\geqslant 0$, $k>0$, there is no intersection of the two curves (13) and (14) other than the origin, when $k<0$, $0<y<\pi/2$, intersections only occur when $x<0$, and when $\pi/2<y<\pi$, they occur only when $x>0$.

Also for $y>0$ and $c>0$, the right hand side of (12) is positive and finite so that

$$\frac{\sin 2y}{\cosh 2x-\cos 2y}$$

must also be positive and finite. Since $\cosh 2x-\cos 2y\geqslant 0$ and $\sin 2y$ is negative within the range $\pi/2<y<\pi$, no zeros can occur in this range. Hence any zeros within the range $0<y<\pi$ must lie within the range $0<y<\pi/2$, and within this range it has been shown that any zeros possess the property $x<0$.

Similarly zeros within the range $n\pi<y<(n+1)\pi$, where n is a positive integer, lie within the restricted range $n\pi<y<(n+\frac{1}{2})\pi$ and possess the property $x<0$.

When $c=0$, it follows immediately from (5) that any solution of (1) is

(*a*) Purely real, but negative, when $b>1$.

(*b*) Complex with a negative real part when $0<b<1$.
(*c*) Negatively infinite when $b=1$.

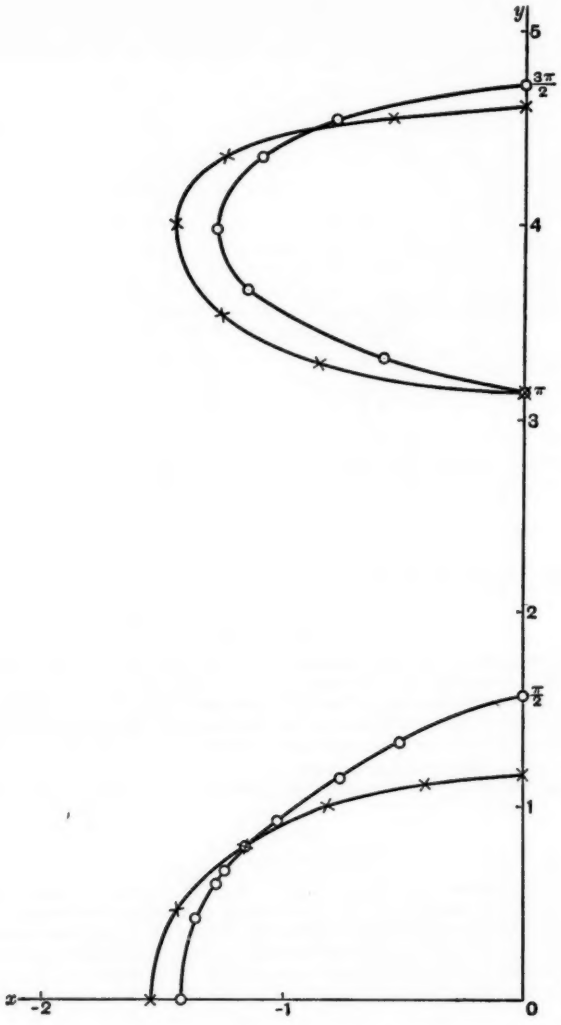

FIG. 2. Graphical determination of the roots of $\coth z = \frac{1}{2z} - \frac{1}{\sqrt{2}}$.

Consequently no complex zeros of equation (1) possess a positive real part when $c<1$ and $b>0$.

The numerical evaluation of the solutions of equation (1) is considered in some detail for the particular case $c=0{\cdot}5$, $b=1/\sqrt{2}$. The points denoted by + in Fig. 2 are the points of intersection of the curves of each side of (11) equated to the same constant, for varying values of that constant. The points denoted by $\odot$ in Fig. 2 have been obtained similarly from (12).

The locus of the points + and $\odot$ in Fig. 2 give respectively the graphs of equations (11) and (12). Each intersection of these two graphs is therefore a solution of (1), with $c=0{\cdot}5$ and $b=1/\sqrt{2}$. Two solutions may be read directly from Fig. 2, namely $x=-1{\cdot}16$, $y=0{\cdot}78$ and $x=-0{\cdot}92$, $y=4{\cdot}47$. However, as the individual points have only been calculated approximately, these two solutions must be regarded as first approximations.

It has been shown above that the solutions of (1) with $b<1$ may be expected to lie in the neighbourhood of $x=-\frac{1}{2}\log_e\dfrac{1+b}{1-b}$, $y=(n+\frac{1}{2})\pi$ for sufficiently large integral values of n.

The method of successive approximations with initial values

$$\left.\begin{matrix} x=-1{\cdot}16 \\ y=\ 0{\cdot}78 \end{matrix}\right\}, \quad \left.\begin{matrix} x=-0{\cdot}92 \\ y=\ 4{\cdot}47 \end{matrix}\right\}, \quad \text{and} \quad \left.\begin{matrix} x=-0{\cdot}881 \\ y=(n+\frac{1}{2})\pi \end{matrix}\right\} n=2(1)9$$

applied to the simultaneous equations (3) and (4) in the form

$$X=-\sinh^{-1}\left\{\left(\frac{-X}{X^2+Y^2}+0{\cdot}7071\right)\Big(\cosh X-\cos Y\Big)\right\},$$

$$Y=\sin^{-1}\left\{\frac{Y}{X^2+Y^2}\Big(\cosh X-\cos Y\Big)\right\},$$

where $X=2x$ and $Y=2y$, gives the solutions of equation (1) within the range $0<y\leqslant 9{\cdot}5\pi$ for the particular case $c=0{\cdot}5$ and $b=1/\sqrt{2}$. These are found to be

x	y
$-1{\cdot}216$	0·693
$-0{\cdot}890$	4·493
$-0{\cdot}885$	7·725
$-0{\cdot}883$	10·904
$-0{\cdot}883$	14·066

x	y
$-0{\cdot}882$	17·220
$-0{\cdot}882$	20·371
$-0{\cdot}882$	23·519
$-0{\cdot}882$	26·666
$-0{\cdot}882$	29·811

The author wishes to express his thanks to the Director of the Electrical Research Association for permission to publish this paper.

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OBLIQUE PEDALS.

By E. H. Neville.

The elementary theorem that, if the triangles ΩAB, ΩPQ are directly similar, so also are the triangles ΩAP, ΩBQ, can be elevated into a general principle :

Given a point Ω and any number of points $P_1, P_2, \ldots$ lying on a curve Γ, let $Q_1, Q_2, \ldots$ be points such that the triangles ΩP_1Q_1, $\Omega P_2Q_2, \ldots$ are all directly similar. Then $Q_1, Q_2, \ldots$ lie on a curve similar to Γ, obtainable by rotating Γ through the angle $P_r\Omega Q_r$ and applying the scale-factor $\Omega Q_r / \Omega P_r$.

This paper makes two applications of this familiar idea.

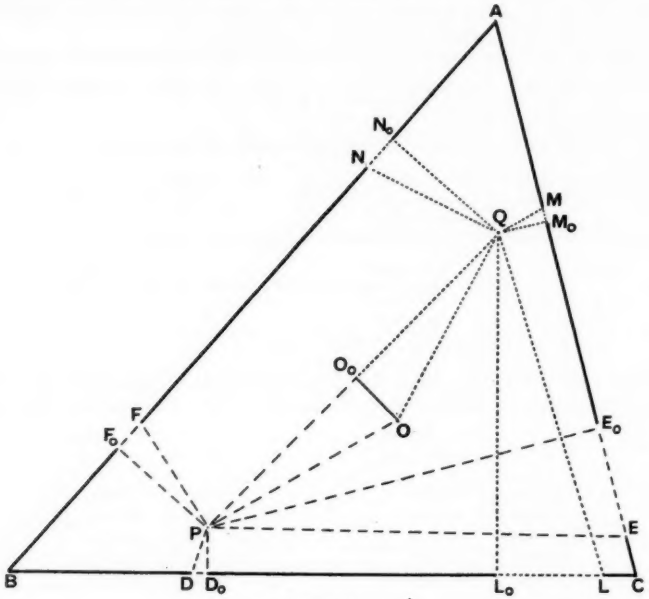


Fig. 1

1. *Isogonal conjugates.*

The principle affords the simplest foundation for the theory of oblique isogonality, recently rediscovered by Mr. Neil Wilson*. The basic theorem may be expressed in the form that

If P, Q are isogonally conjugate in a triangle ABC, and if six points, D and L on a, E and M on b, F and N on c, are such that the six crosses

$$(PD, a),\ (PE, b),\ (PF, c),\ (a, QL),\ (b, QM),\ (c, QN)$$

are all congruent, then these six points are concyclic.

* "The general isogonal circle," *Math. Gaz.*, vol. XXXIV (1950), p. 281. I have adopted or adapted Mr. Wilson's notation for most of the points involved. It is unfortunate that in his figure the angle PFB is obviously obtuse, since the strain on the imagination in equating it to PDC is rendered acute.

This theorem was given by Gallatly* in 1914, without references, in a manner that might indicate that it had been well known for a long time ; it had been published in the previous year by D. F. Barrow†, who regarded it as a lemma for its converse, the theorem which really held his attention.

To put Gallatly's proof in a form in which knowledge of isogonal properties is not assumed, and at the same time to find the centre and radius of the oblique pedal circle, suppose P given, and let Q be the centre of the circle through P_a, P_b, P_c, the images of P in the sides of ABC. Then since the distance of P from the image of Q in any line is equal to the distance of Q from the image of P in that line, P is the centre of the circle through Q_a, Q_b, Q_c, and the two circles have the same radius. Hence, if O_0 is the mid-point of PQ, the circle round O_0 whose radius ρ_0 is half that of the other circles passes through the mid-points of PP_a, PP_b, PP_c, which are the feet D_0, E_0, F_0 of the perpendiculars from P on the sides, and also through the mid-points of QQ_a, QQ_b, QQ_c, which are the feet L_0, M_0, N_0 of the perpendiculars from Q on the sides.

If now the crosses (PD, a), (PE, b), (PF, c) are congruent, the triangles PD_0D, PE_0E, PF_0F are similar, and the circle DEF has centre O and radius $\rho_0 \sec \phi$, where O is the point such that the triangle PO_0O is similar to these three, and ϕ is an angle between PO_0 and PO. But the triangle QO_0O is indirectly congruent with PO_0O. Hence this same circle passes through the three points L, M, N which are such that the triangles QL_0L, QM_0M, QN_0N are indirectly similar to PO_0O, that is, such that the crosses (a, QL), (b, QM), (c, QN) are congruent with (PD, a), (PE, b), (PF, c).

This result implies the usual form of the relation between P and Q as a special case. If E is at A, then A is also one of the two points F, N in which the circle cuts c ; since the crosses (PA, c), (PA, b) can not be congruent, it is only N that can coincide with E at A, and therefore $(c, QA) \equiv (PA, b)$. Similarly, there can be coincidence of F with L at B, and of D with M at C.

Another type of coincidence is possible, not at a vertex. If D and L coincide at a point X, then QX is the reflection of PX in a, and therefore passes through P_a ; that is, X is the point on a where QP_a cuts PQ_a. In the same way, E and M can coincide at the point Y where QP_b cuts PQ_b, and F and N can coincide at the point Z where QP_c cuts PQ_c.

Barrow's theorem can now be proved indirectly. Let a circle cut the sides of ABC in D and L, in E and M, and in F and N. Then the three circles AEF, BFD, CDE are concurrent in a point P which is such that the crosses (PD, a), (PE, b), (PF, c) are congruent. If Q is the isogonal conjugate of P and if the crosses (a, QL'), (b, QM'), (c, QN') are congruent with the other three, the six points D, E, F, L', M', N' all lie on a circle, and since this circle is already determined by the three points D, E, F, the points L', M', N' are identical with L, M, N, and Q is the point of concurrence of the circles AMN, BNL, CLM. Thus

If a circle cuts the sides of the triangle ABC in D and L, in E and M, and in F and N, then the point common to the three circles AEF, BFD, CDE and the point common to the three circles AMN, BNL, CLM constitute an isogonally conjugate pair in the triangle, and the circle is an oblique pedal for this pair.

Since the three pairs of points in which a circle cuts the sides can be divided into complementary triplets in four ways, it follows that

If a circle cuts the sides of a triangle in six distinct points, there are four different isogonally conjugate pairs for which it is an oblique pedal.

* *Modern Geometry of the Triangle*, a booklet which Coolidge does not use ; Gallatly does not distinguish clearly between the theorem and its converse, but is evidently acquainted with both.

† " A theorem about isogonal conjugates," *Amer. Math. Mthly.*, vol. XX, p. 251.

Barrow gives this result explicitly, but Gallatly does not. Mr. Wilson writes as though the interest of his own work lay in the discovery that certain circles are oblique pedals, rather than in the use that can be made of their inevitable pedal character.

Mr. Wilson's extension, to a family of oblique pedals in general, of the generation of Tucker circles by chords of known directions, is important but incomplete. If D_1, M_1 and D_2, M_2 correspond to different values of ϕ with the same isogonal pair P, Q, the points D_2, M_2 occupy homologous positions in the indirectly similar triangles PCD_1, QCM_1; hence, $CD_2/CD_1 = CM_2/CM_1$, and D_2M_2 is parallel to D_1M_1.* That is, with ϕ variable, DM belongs to a definite family of parallel lines. Now let D_1M_1, L_1F_1 cut in U_1' and D_2M_2, L_2F_2 cut in U_2', and let $U_1'U_2'$ cut BC in X'. Then

$$X'D_1/D_1D_2 = X'U_1'/U_1'U_2' = X'L_1/L_1L_2,$$

and therefore $D_1X'/L_1X' = D_1D_2/L_1L_2$. But since the triangles PD_1D_2, QL_1L_2 are indirectly similar, $P_aD_1D_2$, QL_1L_2 are directly similar triangles with their bases on BC, and their centre of similarity is the point X in which BC is cut by P_aQ. That is, $D_1D_2/L_1L_2 = XD_1/XL_1$, and X' coincides with X. In other words, as ϕ_2 varies, U_2' is on a fixed line XU_1', and since when M_2 and F_2 coincide, U_2' coincides with them, this line passes through A. Hence, DM and LF intersect in a point U' which lies on AX, and similarly EN and MD intersect in a point V' which lies on BY, and FL and NE intersect in a point W' which lies on CZ. That is, for varying ϕ, the lines DM, EN, FL are the sides $U'V'$, $V'W'$, $W'U'$ of a triangle whose vertices are on the lines AX, BY, CZ and whose sides have fixed directions. Since for any two values of ϕ the triangles have corresponding sides parallel, the lines joining corresponding vertices, which are the lines AX, BY, CZ, concur in a point R.

This concurrence can be verified directly. Since X is on the line QP_a, $BX/XC = \triangle BP_aQ/\triangle CQP_a$; hence

$$\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB} = \frac{\triangle BP_aQ}{\triangle CQP_a}\cdot\frac{\triangle CP_bQ}{\triangle AQP_b}\cdot\frac{\triangle AP_cQ}{\triangle BQP_c}.$$

But P_b and P_c are equidistant from A and also equidistant from Q, and they do not coincide, for AB, the perpendicular bisector of PP_c, is not the same line as AC, the perpendicular bisector of PP_b. It follows that P_b is the image of P_c in the line AQ, whence $\triangle AQP_b = \triangle AP_cQ$, and Ceva's theorem implies the concurrence of AX, BY, CZ.

It may be noticed that Mr. Wilson's formulae for ρ_0^2, the square of the central radius, follow immediately from the observation that $2\rho_0$ is the length of the side QP_b of the triangle AQP_b. Since $(AC, AP_b) \equiv (AP, AC) \equiv (AB, AQ)$, we have also $(AQ, AP_b) \equiv (AB, AC)$, and A measures an angle from one direction of the line AQ to one direction of the line AP_b. With these directions of measurement,

$$\begin{aligned} 4\rho_0^2 &= AP^2 + AQ^2 - 2AP_b \cdot AQ\cos A \\ &= 2AO_0^2 + \tfrac{1}{2}PQ^2 - 4\triangle AQP_b \cdot \cot A. \end{aligned}$$

An application of this form of the result may be recalled. Since the point

* This argument is preferable to the argument that CD/CM is equal to CP/CQ and is therefore constant, which uses a ratio of lengths that are not parallel. Similarly, Mr. Wilson's expression of the ratio BX/XC as $b.BP.BQ/c.CP.CQ$ can, strictly speaking, prove only that *either* AX, BY, CZ are concurrent *or* X, Y, Z are collinear, and it is a nuisance to have to dispose of the second alternative; that is why I suggest below a different proof.

which is the centre of A, B, C for loads $\tan A$, $\tan B$, $\tan C$ is the orthocentre H, the difference

$$(AO_0^2 \tan A + BO_0^2 \tan B + CO_0^2 \tan C) - HO_0^2(\tan A + \tan B + \tan C)$$

is independent of the position of O_0, and therefore the difference between $HO_0^2 + \frac{1}{4}PQ^2 - 2\rho_0^2$ and $2(\triangle AQP_b + \triangle BQP_c + \triangle CQP_a)$ is the same for all isogonal pairs. Moreover

$$\begin{aligned}
&2(\triangle AQP_b + \triangle BQP_c + \triangle CQP_a)\\
&\quad = (BP_aQ + CQP_a) + (CP_bQ + AQP_b) + (AP_cQ + BQP_c)\\
&\quad = (BP_aC + CQB) + (CP_bA + AQC) + (AP_cB + BQA)\\
&\quad = (CPB + APC + BPA) + (CQB + AQC + BQA)\\
&\quad = 2\triangle ABC,
\end{aligned}$$

and is independent of the positions of P and Q. Hence $HO_0^2 + \frac{1}{4}PQ^2 - 2\rho_0^2$ itself has the same value for all isogonal pairs. If we take P at H, then Q is at the circumcentre O, and since $2\rho_0$ is then the circumradius R, the expression to be evaluated is $\frac{1}{2}(OH^2 - R^2)$, that is, half the power of H for the circumcircle; this is $\frac{1}{2}HA \cdot HH_a$, which is $HA \cdot HD$, where D is the foot of the altitude AH. If the triangle ABC is obtuse-angled, it is self-polar to the circle whose centre is H and the square of whose radius is $HA \cdot HD$; if the triangle is acute-angled, $HA \cdot HD$ is negative and there is no polar circle, but we still call $HA \cdot HD$ the polar constant of the triangle. On the other side $2\rho_0$ is the length of the focal diameter of the conic which has P and Q for foci and touches the sides of the triangle, and if e is the eccentricity of this conic, $\frac{1}{2}PQ^2 = e^2\rho_0^2$; if the conic is an ellipse or an acute hyperbola, $(2 - e^2)\rho_0^2$ is the square of the radius of its orthoptic circle; if $e^2 > 2$, there is no orthoptic circle, but we still call $(2 - e^2)\rho_0^2$ the orthoptic constant of the conic. Thus the equality

$$HO_0^2 + \tfrac{1}{4}PQ^2 - 2\rho_0^2 = \tfrac{1}{2}(OH^2 - R^2)$$

can be read in the form that

If a conic is inscribed in a triangle, the square of the distance between the centre of the conic and the orthocentre of the triangle is the sum of the orthoptic constant of the one and the polar constant of the other.

If the two constants are both positive, this is Gaskin's theorem, that

The orthoptic circle of a conic is orthogonal to the polar circle of a tangent triangle.

Returning to the subject of Mr. Wilson's paper, we remark that the investigation is unchanged if D is associated with N instead of M and L with E instead of F. There is a second triangle $U''V''W''$ whose sides maintain fixed directions if P and Q are given; since the points of coincidence are the same as before, the vertices describe the same lines AX, BY, CZ, and the centre of concurrence is the same point R, but the set of lines through R is different and determines a different circle of the system. It is in fact misleading to speak of *the* circle of concurrence, since two distinct circles have equal right to the name.

It is worth while to observe that the family of oblique pedal circles is determined by the centre of concurrence, and that this point is arbitrary: the lines AR, BR, CR determine the points X, Y, Z, and P and Q are the foci of the one conic which touches the sides of ABC at these points.

2. *A note on conics.*

From the fundamental principle enunciated at the beginning of this article, the oblique pedals of a curve for a given point are similar curves, each obtain-

able by rotation and enlargement from the pedal as defined normally. In particular, if Z_ϕ is the point on the variable tangent q to a conic such that the cross (SZ_ϕ, q) is always congruent with some fixed cross, the locus of Z_ϕ is a circle,* whose centre C_ϕ is the point such that SCC_ϕ is congruent with SZ_0Z_ϕ, where Z_0 is the foot of the perpendicular from S on q. Since the angle at Z_0 is a right angle, the centre C_ϕ is on the secondary axis of the conic, a line which we will denote by b.

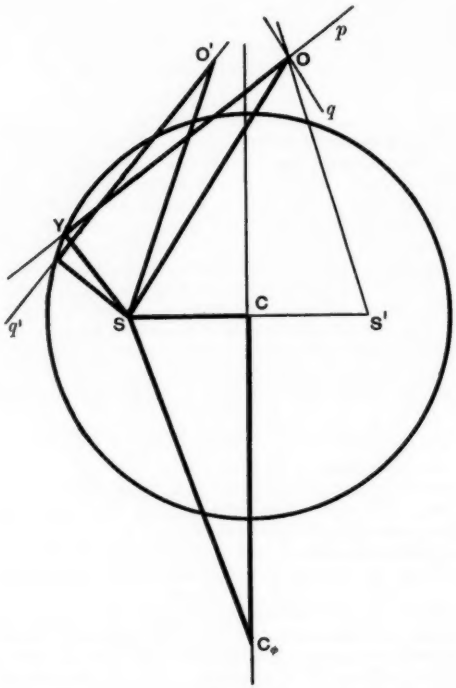


FIG. 2

Now let P be a fixed point of the conic, and consider the oblique pedal determined by the congruence $(SZ_\phi, q) \equiv (SP, p)$. This locus includes the point P, and since the centre of the locus is on b, includes also the image P' of P in b. Since P' is on the conic, the only tangent *through* P' is the tangent p' *at* P', and therefore $(SP', p') \equiv (SP, p)$. From symmetry about b, (SP', p') is indirectly congruent with $(S'P, p)$. Hence $(S'P, p)$ is indirectly congruent with (SP, p), that is, one focal radius is the reflection of the other in the tangent.

* To anticipate a charge of circular reasoning, let me remind readers that to prove that, in the usual notation, the locus of Y is the auxiliary circle, we have only to show that the triangle SYX is similar to SPM.

The interest of this proof of the familiar theorem is that the argument throughout is about properties of the tangent itself. We can repeat the argument, taking an arbitrary point O on the tangent p and determining the pedal by the congruence $(SZ_\phi, q) \equiv (SO, p)$. Since the image O' of O in b is on this pedal, the line q' through O' given by $(SO', q') \equiv (SO, p)$ is a tangent to the conic, and from symmetry it follows that the line q through O given by $(q, S'O) \equiv (SO, p)$ is a tangent also. But if O is not at P, the line $S'O$, since it does not pass through the image of S in p, is not the reflection of SO in p, and therefore a line q satisfying $(q, S'O) \equiv (SO, p)$ can not be the line p. Thus q is the second tangent through O, and the isogonal relation between tangents and focal radii is established.

If C_ϕ is the centre of the circle determined by $(SZ_\phi, q) \equiv (SP, p)$, the triangle SCC_ϕ is similar to SYP, and therefore the triangle SPC_ϕ is similar to SYC, and in particular $(SP, PC_\phi) \equiv (SY, YC)$. But YC, being parallel to $S'P$, is parallel to the reflection of SP in any line parallel to the normal PG; that is, $(SY, YC) \equiv (SP, PG)$. Hence PC_ϕ is the normal, and the oblique pedal touches the conic at P, and therefore also at P'. *The circle which touches the conic at the ends of any chord parallel to the focal axis is an oblique pedal.* The converse is true if the conic is a hyperbola; for an ellipse, if the sine of the acute angle between SZ_ϕ and q is less than b/a, the oblique pedal is a circle which has no points in common with the conic. The circles which have double contact at the ends of chords parallel to the secondary axis are not, in real geometry, pedals at all. Analytically the two forms

$$(Ax^2 + By^2 - 1) + (A - B)(y - k)^2 = 0, \quad (Ax^2 + By^2 - 1) + (B - A)(x - h)^2 = 0$$

are indistinguishable, and these cover all circles having double contact with $Ax^2 + By^2 = 1$. E. H. N.

GLEANINGS FAR AND NEAR

1781. It has been remarked that the study of an analytic function

$$w = f(z) = u + jv,$$

of a complex variable $z = x + jy$, amounts essentially to the study of a pair of real functions $u(x, y)$ and $v(x, y)$ which satisfy the Cauchy-Riemann equations. While in a sense this is true the student should not get the impression that the quantity $\sqrt{(-1)}$ is introduced purely as a convenient and shorthand method of notation. That would be analogous to believing that one can describe a beautiful girl in terms of her weight and dimensions and other measurable attributes, while ignoring the characteristics of the girl which are due to the overall combination.

Footnote. The equation $e^{j\pi} = -1$ has been called the eutectic point in mathematics. This is a very appropriate metaphor for no matter how you boil down and explain this equation which relates four of the most remarkable numbers in mathematics it still has a certain mystery about it which cannot be explained away.—S. Goldman, *Transformation calculus and electrical transients* (1949), pp. 186–7. [Per Mr. H. V. Lowry.]

1782. With some thin wire, make a " cage " in the form of a regular tetrahedron. . . . If $ABCD$ is the cage, and AX is perpendicular to BCD, O, which is a quarter of the way up XA, is the centre of symmetry.—*Mathematical Pie* (1953), **10**, 69. [Per Mr. G. N. Copley.]

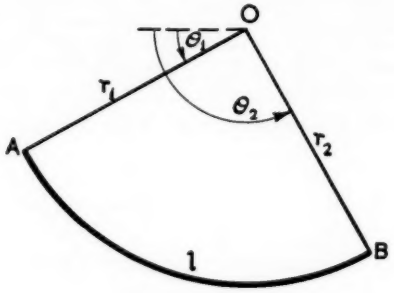
EULER'S EQUATION AND (p, r) COORDINATES

By K. E. Bullen

1. In a seismological context,* it was recently noticed that the use of (p, r) coordinates in a variation problem where polar coordinates had previously been used presented some unusual features. The method followed is one of some generality, and the purpose of the present note is to illustrate it in a suitable elementary problem : " A heavy uniform inelastic string has its ends fastened to two assigned points on the surface of a smooth right circular cone fixed with its axis vertical and vertex O uppermost, and rests in equilibrium on the surface of the cone. Prove that if this surface is developed into a plane, the curve on which the string lay is given by the form $p(a+br)=1$."

This problem has appeared in various examination papers with the hint that polar coordinates in the developed plane should be used, and the solution by this route will be outlined in Section 2. The alternative solution in which (p, r) coordinates are used exlusively will be discussed in Sections 3–7.

2. *Solution using polar coordinates.* In the developed plane, let (r, θ) be polar coordinates referred to O as origin ; and let (r_1, θ_1), (r_2, θ_2) be the co-



ordinates of the terminal points A, B of the string. The accompanying figure relates to the developed plane.

The potential energy V of the string is seen to be given by

$$V = k\int_{\theta_1}^{\theta_2} r\sqrt{(r^2+r'^2)}\,d\theta, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where k is an assigned constant, and $r' \equiv dr/d\theta$. The integral in (1) has to be stationary, subject to the relation

$$\int_{\theta_1}^{\theta_2} \sqrt{(r^2+r'^2)}\,d\theta = l, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where l is the constant length of the string.

Following the usual method for solving isoperimetrical problems,† it is then necessary that

$$\int_{\theta_1}^{\theta_2} \phi\,d\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

should be stationary, where $\phi = (r+\lambda)\sqrt{(r^2+r'^2)}$, and λ is an undetermined

* K. E. Bullen, " Parameters of seismic rays ", *Proc. of Roy. Soc. of N.S.W.*, 1953.

† See e.g. Byerly, *Introduction to the Calculus of Variations*, Harvard, §10, 1920.

multiplier. Since θ does not occur explicitly in ϕ, the resulting Euler equation yields the first integral $\phi - r' \partial\phi/\partial r' = C$, where C is constant; and this on reduction yields

$$(r+\lambda)r^2 = C\sqrt{(r^2+r'^2)}. \quad \text{.....(4)}$$

On now changing from (r, θ) to (p, r) coordinates, we derive the equation

$$p(r+\lambda) = C, \quad \text{.....(5)}$$

which is equivalent to the stated relation

$$p(a+br) = 1. \quad \text{.....(6)}$$

The constants (λ, C) and hence (a, b), can be formally determined from (4), using the facts that the length l of the string is assigned, and that

$$r = r_1, r_2 \text{ at } \theta = \theta_1, \theta_2, \quad \text{.....(7)}$$

where $r_1, r_2, \theta_1, \theta_2$ are assigned values.

3. *Exclusive use of (p, r) coordinates.* If (p, r) coordinates are used from the outset, we write

$$V = k\int_{r_1}^{r_2} r^2(r^2-p^2)^{-\frac{1}{2}}\,dr, \quad \text{.....(8)}$$

$$\int_{r_1}^{r_2} r(r^2-p^2)^{-\frac{1}{2}}\,dr = l, \quad \text{.....(9)}$$

in place of (1), (2) of Section 2.

The equation formed analogously to (3) of Section 2 would be

$$\int_{r_1}^{r_2} \psi\,dr, \quad \text{.....(10)}$$

where $\psi = (r^2+\lambda r)/\sqrt{(r^2-p^2)}$. The Euler equation resulting from the condition for (10) to be stationary is

$$\frac{\partial\psi}{\partial p} - \frac{d}{dr}\frac{\partial\psi}{\partial p'} = 0, \quad \text{.....(11)}$$

where $p' \equiv dp/dr$.

We immediately notice an important difference between ψ and ϕ in that ψ does not contain the derivative of p, whereas ϕ contains the derivative r' of the dependent variable r in Section 2. Thus (11) becomes simply $\partial\psi/\partial p = 0$. This is not a differential equation, and reduces to

$$p(r^2+\lambda r) = 0. \quad \text{.....(12)}$$

Now (12) is incompatible with (6). Furthermore, because of the unusual feature just referred to, (12) does not contain two constants. It may now be asked: (i) Why is there a discrepancy between the number of constants in (6) and (12)? (ii) Where is the fault in the argument in Section 3? (iii) How is the valid solution derived using only (p, r) coordinates?

4. The answer to (i) is of course to be found in the special property of the coordinate p which is connected with r, θ by a differential relation, and not by an algebraic relation in which all constants are known.

5. The key to (ii) lies in the set of conditions (7). Where, as in Section 2, the values of both independent and dependent variables are assigned at the terminal points, the conditions (7) were automatically taken care of in the procedure followed. In the case of (p, r) coordinates, this applies only to the first pair of conditions in (7) since the independent variable r is assigned at A and B, while p is not, and the condition that θ is assigned at A and B has not been incorporated. It is therefore necessary to express in terms of p and r the

fact that the angle AOB is equal to an assigned constant, α say. Hence the relation

$$\int_{r_1}^{r_2} pr^{-1}(r^2-p^2)^{-\frac{1}{2}}\,dr=\alpha \qquad \text{.....(13)}$$

has to be set down as a further restriction on the connection of p with r, and one which involves an integral in the same way as (9) does.

The enunciated problem is thus adequately represented by (8), (9), (13) and the fact that r is fixed at A and B. Without (13), the representation is inadequate.

6. We can now deal with (iii) and complete the solution. In place of (10), we take

$$\int_{r_2}^{r_1}\eta\,dr, \qquad \text{.....(14)}$$

as the integral to be made stationary, where

$$\eta=(r^2+\lambda r+\mu pr^{-1})/\sqrt{(r^2-p^2)}, \qquad \text{.....(15)}$$

μ being a *second* undetermined multiplier, corresponding to the presence of the second equation of condition (13).

We now have (11) with ψ replaced by η, and, as before, the second term on the left-hand side is zero, leaving us with $\partial\eta/\partial p=0$. Substituting from (15) and simplifying, we obtain

$$(r+\lambda)p+\mu=0,$$

which is equivalent to (6); and the problem is solved.

It needs to be remarked that η as given by (15) is infinite if the curve includes a point at which $p=r$, but the argument is readily adapted to cover this case by splitting the integral in (14) into two parts, and the result is not affected.

7. The discussion in Sections 5 and 6 first of all focuses attention on the importance of attending properly to all boundary conditions before applying routine processes in the Calculus of Variations.

The really interesting point which then emerges is that the use of (p, r) coordinates has reduced the problem to one in which the Euler equation in the full form (11) is not needed. In making (14) stationary, it was simply necessary to differentiate the integrand with respect to the dependent variable p, the use of undetermined multipliers leading directly to the solution without any integration being necessary.

A method which avoids the Euler differential equation in this way can shorten the solutions of a variety of problems on stationary integrals. Other coordinates than the pair (p, r) can also be used to advantage, and in some cases the saving of labour is very considerable. Applications to a number of elementary standard problems will be presented in a later note.

K. E. B.

1783. You will be pleased to hear that I did three problems in geometry yesterday without assistance. Mr. Keith and Teacher were quite enthusiastic over the achievement, and I must confess I felt somewhat elated myself. Now I feel as if I should succeed in doing something in mathematics, although I cannot see why it is so very important to know that the lines drawn from the extremities of the base of an isosceles triangle to the middle points of the opposite sides are equal. The knowledge doesn't make life any sweeter or happier, does it? On the other hand, when we learn a new word, it is the key to untold treasures.—Helen Keller, *Story of my Life*, (1953), p. 202. [Per Rev. A. F. Mackenzie.]

NOTES ON DIFFERENTIAL EQUATIONS

By F. Underwood.

These notes deal with substitutions or changes of variables in different types of differential equations. For convenience they are classified under the heading of each type of differential equation where they are used.

I. Charpit's equations.

If $p=\partial z/\partial x$, $q=\partial z/\partial y$, it will be noted that many partial differential equations are found in which p always occurs in the form x^mp and q in the form y^nq, where m and n are constants. In some cases the equation may not be found in this form at first, but can be rewritten in this form.

(*a*) If $m\neq 1$ and $n\neq 1$, the substitution is $X=x^{1-m}$, $Y=y^{1-n}$.

Then
$$P=\frac{\partial z}{\partial X}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dX}=\frac{x^mp}{1-m},\quad Q=\frac{\partial z}{\partial Y}=\frac{y^nq}{1-n}.$$

(*b*) If $m=1$ or $n=1$, the corresponding substitutions are $X=\log x$, $Y=\log y$, and then $P=xp$, $Q=yq$.

If the equation contains both p and q in the forms z^kp, z^kq, or can be reduced to such a form, where k is constant, similar substitutions can be used.

(*c*) If $k\neq -1$, let $Z=z^{k+1}$.

Then
$$P=\frac{\partial Z}{\partial x}=\frac{dZ}{dz}\cdot\frac{\partial z}{\partial x}=(k+1)z^kp,\quad Q=\frac{\partial Z}{\partial y}=(k+1)z^kq.$$

(*d*) If $k=-1$, let $Z=\log z$. Then $P=p/z$, $Q=q/z$.

By the use of the appropriate substitutions chosen from (*a*), (*b*), (*c*), (*d*) many equations which would normally require the use of Charpit's equations are reduced to the simple forms usually known as Standards I, II, III [F., 407–414, or P., 153–154].* Detailed references are given by F. H. Miller [M., 118–120]. The same substitutions are useful for other equations, though it may not be possible to reduce these equations to Standards I, II or III.

Simple examples which may be treated by the above substitutions are :

(i) $x^2p^2+ayzq=bz^2$. [K., II, 191, No. 17].

Writing this as $(xp/z)^2+a(yq/z)=b$, then $m=n=1$, $k=-1$. If we use only the substitutions (*b*), the equation becomes

$$P^2+azQ=bz^2,$$

which is of Standard II. If, however, we use both (*b*) and (*d*), (i) is reduced to

$$P^2+aQ=b,$$

which is of Standard I. If $P=c$, then $Q=(b-c^2)/a$ and then $dZ=P\,dX+Q\,dY$ gives immediately

$$aZ+h=acX+(b-c^2)Y$$

or
$$a\log z+h=ac\log x+(b-c^2)\log y:$$

that is,
$$Az^a=x^{ac}y^{b-c^2}.$$

(ii) $x^2y^2z^2pq^2=1$ [J., 324, No. 15].

This may be written $(x^2zp)(y^2z^2q)^2=1$, so that $m=2$, $n=3/2$, $k=1$. The substitutions (*a*) and (*c*) reduce this to

$$PQ^2+32=0,$$

* References are given at the end of this article.

which is of Standard I, so $Q=4a$ gives $P=-2/a^2$ and then $dZ=P\,dX+Q\,dY$ gives

$$Z=-2X/a^2+4aY+b$$

or

$$z=-2/a^2x+4a/\sqrt{y}+b.$$

It is interesting to compare this with the use of Charpit's equations for $F=x^2y^2z^2pq^2-1=0$. From

$$\frac{dx}{-\frac{\partial F}{\partial p}}=\frac{dy}{-\frac{\partial F}{\partial q}}=\frac{dz}{-p\frac{\partial F}{\partial p}-q\frac{\partial F}{\partial q}}=\frac{dp}{\frac{\partial F}{\partial x}+p\frac{\partial F}{\partial z}}=\frac{dq}{\frac{\partial F}{\partial y}+q\frac{\partial F}{\partial z}}$$

we have

$$\frac{dx}{-x^2y^2z^2q^2}=\frac{dy}{-2x^2y^2z^2pq}=\frac{dz}{-x^2y^2z^2pq^2-2x^2y^2z^2pq^2}=\frac{dz}{-3}$$
$$=\frac{dp}{2xy^2z^2pq^2+3x^2y^2z^2p^2q^2}=\frac{dq}{3x^2y^2z^2pq^2+3x^2y^2z^2pq^3}.$$

II. *Jacobi's equations.*

Substitutions of the type under I above may be applied in certain cases to equations usually treated directly by Jacobi's method. As there are now more than two independent variables, m, n are replaced by m_1, m_2, m_3, ... , but k remains as in I. This may be illustrated by two simple examples.

(iii) $p_1p_2p_3=z^3x_1x_2x_3$. [P., 168, No. 5].

Writing this as $(p_1/zx_1)(p_2/zx_2)(p_3/zx_3)=1$, then $m_1=m_2=m_3=-1$, $k=-1$. Thus the substitutions

$$X_r=x_r^2(r=1,2,3)\,;\quad Z=\log z$$

give

$$8P_1P_2P_3=1,$$

which may be regarded as analogous to Standard I in the earlier work. Taking $2P_1=a$, $2P_2=b$, $2P_3=1/ab$, we have

$$2Z=aX_1+bX_2+(X_3/ab)+c,$$

or

$$2\log z=ax_1^2+bx_2^2+(x_3^2/ab)+c.$$

The following example shows the corresponding reduction for a pair of equations possessing a common integral.

(iv) $\left.\begin{aligned}x_2p_1^2-p_2+2x_2x_3^2p_3=0\\ x_2p_1+p_3=0\end{aligned}\right\}$ [M., 129, No. 2].

These may be written as $p_1^2-p_2/x_2+2x_3^2p_3=0$ and $p_1+p_2/x_2=0$ respectively. Hence $m_1=0$, $m_2=-1$, $m_3=2$, $k=0$, giving $X_2=x_2^2$, $X_3=1/x_3$, and the equations become

$$F=p_1^2-2P_2-2P_3=0,\quad G=p_1+2P_2=0.$$

Since these equations do not contain x_1, X_2, X_3 or z, it is obvious that they satisfy the Poisson Bracket condition $(F,\,G)=0$. Putting $p_1=2a$, we have $P_2=-a$, $P_3=2a^2+a$, and hence

$$dz=2a\,dx_1-a\,dX_2+(2a^2+a)\,dX_3$$

and the common integral is

$$z=2ax_1-aX_2+(2a^2+a)X_3+b$$

or

$$z=2ax_1-ax_2^2+(2a^2+a)/x_3+b.$$

III. *Monge's method*

In this case we are not concerned with substitutions in the first part of the process, but can use what may be regarded as a general class of substitutions for the final stage of integration. In Monge's method the first stage usually ends with the formation of two intermediate integrals of the forms

$$f(p, q) = \phi(u), \quad F(p, q) = \psi(v),$$

where u and v are functions of x, y, z only, though sometimes only one intermediate integral is obtained. If these equations are solved for p and q, the solutions are of the forms

$$p = \phi_1(u, v), \quad q = \psi_1(u, v),$$

where ϕ_1, ψ_1 involve ϕ and ψ as functions of u, v respectively. The final integration is then simplified by the use of new variables u, v, w where w is one of the original variables x, y, z. One example is sufficient to illustrate the process.

(v) $r(1-q^2) + s(2pq + q - 1) - tp(p+1) = 0$. [London M.A. (External) 1933].

Monge's method gives immediately the two intermediate integrals

$$(q+1)/(p+1) = \phi(z-y) \quad \ldots\ldots(\alpha)$$

$$(q-1)/p = \psi(x+y+z). \quad \ldots\ldots(\beta)$$

Then $p = (2-\phi)/(\phi - \psi), \quad q = (\phi + \psi - \phi\psi)/(\phi - \psi).$

Hence $dz = p\,dx + q\,dy$ gives

$$(\phi - \psi)\,dz = (2-\phi)\,dx + (\phi + \psi - \phi\psi)\,dy. \quad \ldots\ldots(\gamma)$$

Let $x + y + z = u, \quad z - y = v,$

so that $x + 2z = u + v, \quad x + 2y = u - v.$

Then $2\,dz = du + dv - dx, \quad 2\,dy = du - dv - dx.$

Now (γ) gives

$$(\phi - \psi)(du + dv - dx) = (4 - 2\phi)dx + (\phi + \psi - \phi\psi)(du - dv - dx).$$

Hence $(2-\phi)(2-\psi)dx + \psi(2-\phi)du - \phi(2-\psi)dv = 0,$

or $$dx + \frac{\psi\,du}{2-\psi} - \frac{\phi\,dv}{2-\phi} = 0.$$

Since ψ is a function of u only and ϕ is a function of v only,

$$x = f(u) + F(v) = f(x+y+z) + F(z-y).$$

IV. *The total differential equation* $P\,dx + Q\,dy + R\,dz = 0$.

In many cases the integration of this equation can be simplified by change of variables so that x, y, z are replaced by u, v, w, where u, v, w, are all functions of x, y, z which satisfy the condition

$$J = \frac{\partial(u, v, w)}{\partial(x, y, z)} \neq 0.$$

(*a*) *The homogeneous equation*

If P, Q, R are homogeneous functions of the same degree in x, y, z, and if the condition of integrability is satisfied, then an integrating factor of this equation is $1/E$, where $E = Px + Qy + Rz$, provided that $E \neq 0$. If this condition is satisfied, it frequently happens that E is of the form $E = uvw\ldots$, where u, v, w are simple polynomials in x, y, z. In many cases the integration following the

use of the factor $1/E$ is obvious, but in others the integration is given more readily by the use of new variables u, v, w, where one or two of these variables may be identical with the original variables. If this course is followed, the substitutions can be made in the original equation.

(vi) $$(2x-y-z)\,dx+(x-2y+z)\,dy-3(x-y)\,dz=0.$$

$$E=Px+Qy+Rz=x(2x-y-z)+y(x-2y+z)-3z(x-y)$$
$$=2(x-y)(x+y-2z).$$

Let $$u=x-y, \qquad v=x+y-2z, \qquad w=z.$$

Then $$2(x-z)=u+v, \qquad 2(y-z)=v-u;$$
$$2\,dx=du+dv+2\,dz, \qquad 2\,dy=dv-du+2\,dz.$$

The equation now reduces to

$$du/u+3\,dv/v=0$$

and the integral is

$$uv^3=(x-y)(x+y-2z)^3=A.$$

The next example gives a simple case in which the equation is not homogeneous at first, but may be made so by elementary substitutions.

(vii) $2x(y^2-z)\,dx+(y^2+z-2x^2y)\,dy+(x^2-y)\,dz=0$. [N.M., 104, No., 15].

Let $$x^2=X, \qquad y=Y, \qquad z=Z^2.$$

Then $$(Y^2-Z^2)\,dX+(Y^2+Z^2-2XY)\,dY+2Z(X-Y)\,dZ=0,$$

and $$E=-(X-Y)(Y^2-Z^2).$$

Let $$u=X-Y, \qquad v=Y^2-Z^2, \qquad w=Y;$$
$$dX=du+dY, \qquad 2Z\,dZ=2Y\,dY-dv.$$

We then have $v\,du-u\,dv=0$, so that $u=Av$, and

$$x^2-y=A(y^2-z).$$

(viii) $(x^2+2xy+y^2-z^2)\,dx-(x^2+2xy+y^2+z^2)\,dy+2(x+y)z\,dz=0$.
[London (Internal), 1937].

Here $E=(x+y)(x^2-y^2+z^2)$ and the substitutions

$$u=x+y, \qquad v=x^2-y^2+z^2, \qquad w=z$$

reduce the equation to the form $u\,dv-v\,du=0$, and give the integral

$$x^2-y^2+z^2=A(x+y).$$

This example may also be used to illustrate cases in which the new functions u, v, w are not all factors of E. Thus

$$u=x+y, \qquad v=x-y, \qquad w=z$$

gives $$-z^2\,du+u^2\,dv+2uz\,dz=0$$

or $$dv+2z\,dz/u-z^2\,du/u^2=0.$$

Then $$v+z^2/u=A,$$

and thence $x^2-y^2+z^2=A(x+y)$, as above.

(ix) $(2z^2-xy+y^2)z\,dx+(2z^2+x^2-xy)z\,dy-(x+y)(xy+z^2)\,dz=0$.
[F., 573, No. 70 (ix)].

On reduction $$E=z(x+y)(z^2-xy).$$

Let $$u=x+y, \qquad v=z^2-xy, \qquad w=z.$$

Then $(x-y)\,dx = x\,du + dv - 2z\,dz, \qquad (x-y)\,dy = 2z\,dz - y\,du - dv.$

Equation (ix) reduces to

$$2zv\,du - zu\,dv + uv\,dz = 0$$

or
$$2\,du/u - dv/v + dz/z = 0.$$

Hence
$$u^2z = Av,$$

or
$$z(x+y)^2 = A(z^2 - xy).$$

(*b*) *Non-homogeneous equations.*

In certain cases equations which are not homogeneous are of such forms that substitutions of the types used above are suggested by the equations. No general rule or method can be given here, for the substitutions used obviously depend upon the form of the particular equation under consideration. Thus (vii) above might have been solved with using a homogeneous form by the substitutions

$$u = x^2 - y, \qquad v = y^2 - z, \qquad w = y.$$

Another example in which this method of elementary substitutions is useful is :

(x) $(2y + z + x^2y^3 + x^2y^2z)dx + (2x + z + x^2y^2z + x^3y^2)dy + (x+y)(1 + x^2y^2)\,dz = 0.$ [London (Internal), 1943].

With
$$u = xy, \qquad v = x + y, \qquad w = z,$$

we have
$$(x-y)\,dx = x\,dv - du, \qquad (x-y)\,dy = du - y\,dv,$$

and the equation reduces to

$$(v\,dz + z\,dv) + du + du/(1+u^2) = 0.$$

Thus $vz + u + \tan^{-1}u = yz + zx + xy + \tan^{-1}xy = A.$

An example which may be used to illustrate the variety in manipulation associated with different choices of the new variables is :

(xi) $(1+yz)\,dx + x(z-x)\,dy - (1+xy)\,dz = 0.$ [Cambridge, 1947].

Consider as possible new variables

$$u = 1 + yz, \qquad v = 1 + xy, \qquad w = z - x.$$

(*a*) The system v, w, x reduces (xi) to $w\,dv - v\,dw = 0$ and gives the integral

$$1 + xy = A(z - x).$$

(*b*) The system u, v, y gives

$$y(u\,dv - v\,du) + v(u-v)\,dy = 0$$

which can be written

$$dy/y + dv/v = (du - dv)/(u - v).$$

The integral is

$$yv = A(u - v)$$

or $1 + xy = A(z - x)$, as before.

(*c*) The system u, w, z, after some reduction, gives

$$wz(z-w)\,du - uz^2\,dw + (u-1)w^2\,dz = 0$$

which is apparently as difficult as the original equation, but, on division by w^2z^2, can be written

$$\left\{\frac{du}{w} - \frac{u\,dw}{w^2}\right\} - \left\{\frac{du}{z} - \frac{(u-1)\,dz}{z^2}\right\} = 0.$$

Hence
$$\frac{u}{w}-\frac{u-1}{z}=A,$$
from which again we obtain
$$1+xy=A(z-x).$$

(*d*) The system u, v, w gives
$$x=w(v-1)/(u-v), \quad y=(u-v)/w, \quad z=w(u-1)/(u-v).$$
Hence
$$dx=\frac{w\,dv+(v-1)\,dw}{u-v}-\frac{w(v-1)(du-dv)}{(u-v)^2},$$
$$dy=\frac{du-dv}{w}-\frac{(u-v)\,dw}{w^2},$$
$$dz=\frac{w\,du+(u-1)\,dw}{u-v}-\frac{w(u-1)(du-dv)}{(u-v)^2}.$$
Equation (xi) then reduces to $w\,dv-v\,dw=0$, as in (*a*).

In all cases in these notes simple examples have been chosen to illustrate the use of substitutions, but this does not imply that the method used is necessarily the shortest and most direct for a particular equation.

F. U.

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1784. Wallis seems to have shown no marked interest in mathematics until he was some fifteen years old. Then, while he was on holiday from Felsted, he chanced upon his brother's arithmetic book. Curious to know the meaning of its odd signs and symbols, he borrowed the book and mastered it within a fortnight. Thereafter mathematics was to be his principal interest in life. His outstanding work was on the geometric treatment of *imaginary numbers, such as minus quantities.—Manchester Guardian*, 28th October, 1953. [Per Dr. B. H. Neumann.]

1785. Headmasters estimate that if your child is at a private or public school, the coming term will cost you 40 per cent. more than it would have done five years ago. Here is how that figure is made up. Rise in school fees is, in general, 18 per cent., rise in the cost of school clothes, 10 per cent., in the cost of books and school equipment, 12 per cent. The last figure would be higher still but for the fact that the Book Publishers Association have deliberately kept the price of text-books down.—*Housewife*, September, 1952. [Per Mr. B. D. Price.]

UNITS AND DIMENSIONS.

By H. V. Lowry.

A new method of dealing with units and dimensions, in particular with the relationship between the various systems of electrical units including the M.K.S. system.

1. It is usual when discussing units and dimensions for such an equation as $F=kma$ to limit the discussion to systems in which $k=1$. It is, however, far easier in discussing electrical units to leave such constants in the dimensional formulae because we can then carry out transformations from one set of units to another more easily and, when we know the constant for one system of units we can immediately find its value for any other and so we are enabled not only to check the dimensions of a formula but, *at the same time, to find the constant*. To explain this briefly for the case of the formula $F=kma$, suppose $[F]$, $[m]$, $[a]$ are the units of force, mass and acceleration, then, if a particular force give a particular mass an acceleration, the products $F[F]$, $m[m]$, $a[a]$ each have the same value for all unit systems. Consequently

$$\frac{[F]}{[m][a]}=\frac{am}{F}\times\text{a constant}=\frac{B}{k},\text{ say.}$$

and hence $[F]$ is proportional to $k^{-1}[m][a]$, where k is a dimensionless number. To show how this result may be used, consider the problem of finding k when $[F]=1$ ton wt., $[m]=1$ oz., and $[a]=1$ ml./hr.^{2}. We know that $k=1$ when $[F]=1$ poundal, $[m]=1$ lb., $[a]=1$ ft./sec.^{2}. Hence

$$\frac{1\text{ ton wt.}}{1\text{ poundal}}=\frac{k^{-1}}{1}\frac{1\text{ oz.}}{1\text{ lb.}}\frac{1\text{ ml./hr}^2}{1\text{ ft./sec.}^2};$$

whence

$$k=\frac{1}{32\times2240}\cdot\frac{1}{16}\cdot\frac{5280}{3600^2}.$$

2. *Electrical units.*

In the equations of electricity it is essential to introduce constants dependent on the system of units into some of the equations. It is convenient not to introduce such constants into the following equations, and this is the usual practice. The notations used are those in standard use.

$$E=F/q,\quad H=F/m,\quad i=dq/dt,\quad j=i/A,$$
$$\operatorname{div}\mathbf{j}+\partial\rho/\partial t=0,\quad \phi=q/C=\int\mathbf{E}\cdot\mathbf{ds}=Ri,$$
$$Li=\int\mathbf{B}\cdot\mathbf{dS}.$$

It is recommended here that constants $\gamma_1,\ldots,\gamma_4$ should be introduced into the other equations as follows

$$\frac{\gamma_1 q^2}{Kr^2}=F,\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

$$\frac{\gamma_2 m^2}{\mu r^2}=F,\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

$$\left.\begin{aligned}\int\mathbf{H}\cdot\mathbf{ds}&=\frac{4\pi\gamma_3}{c^2}\int(\mathbf{j}+\mathbf{j}')\cdot\mathbf{dS}\\ \text{or}\qquad\operatorname{curl}\mathbf{H}&=\frac{4\pi\gamma_3}{c^2}(\mathbf{j}+\mathbf{j}')\end{aligned}\right\},\quad\ldots\ldots\ldots\ldots\text{(iii)}$$

$$\left.\begin{aligned} \int \mathbf{E}\cdot d\mathbf{s} &= -\frac{\gamma_4}{c^1}\int \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S} \\ \text{or}\qquad \operatorname{curl}\mathbf{E} &= -\frac{\gamma_4}{c^1}\frac{\partial \mathbf{B}}{\partial t} \end{aligned}\right\}, \quad \text{.....(iv)}$$

$$\mathbf{D} = \gamma_5 K\mathbf{E}, \quad \text{.....(v)}$$

$$\mathbf{B} = \gamma_6\mu\mathbf{H}. \quad \text{.....(vi)}$$

For these equations K, μ are to be taken as 1 in a vacuum and so to be independent of the unit system. c^1 is the velocity of light, the suffix 0 being used to distinguish it from c, which will be taken to be the measure of c^1, thus c^1 has dimensions, but c has not. The velocity of light has been put in equations (iii) and (iv) straight away because it is found that, if we do this, the six γ's can all be non-dimensional constants. $\mathbf{j}'$ is the density of the displacement current.

Since div curl $\mathbf{H} = 0$, we have div $(\mathbf{j} + \mathbf{j}') = 0$, but from (i) and (v),

$$\operatorname{div}\mathbf{D} = 4\pi\gamma_1\gamma_5\rho \quad \text{.....(vii)}$$

whence, using div $\mathbf{j} + \partial\rho/\partial t = 0$, we have

$$\operatorname{div}\left(\mathbf{j} + \frac{1}{4\pi\gamma_1\gamma_5}\frac{\partial \mathbf{D}}{\partial t}\right) = 0$$

and so we identify $\mathbf{j}'$ with $\dfrac{1}{4\pi\gamma_1\gamma_5}\dfrac{\partial \mathbf{D}}{\partial t}$, and then (iii) can be written

$$\operatorname{curl}\mathbf{H} = \frac{4\pi\gamma_3}{c^1}\left(\mathbf{j} + \frac{1}{4\pi\gamma_1\gamma_5}\frac{\partial \mathbf{D}}{\partial t}\right). \quad \text{.....(iii}a\text{)}$$

From the formulae which do not contain gammas and from formulae (i) to (vii) we find quite easily that the units of the electrical quantities are proportional to the expressions in the right-hand column in the table below, in which l, T, F (without brackets for convenience here) stand for the units of length, time and force.

$[q]$	$\gamma_1^{\frac{1}{2}} l F^{\frac{1}{2}}$		$[R]$	$\gamma_1^{-1} l^{-1} T$	
$[m]$	$\gamma_2^{\frac{1}{2}} l F^{\frac{1}{2}}$		$[H]$	$\gamma_2^{-\frac{1}{2}} l^{-1} F^{\frac{1}{2}}$,	from (ii)
$[E]$	$\gamma_1^{-\frac{1}{2}} l^{-1} F^{\frac{1}{2}}$	or	$[H]$	$\gamma_1^{\frac{1}{2}}\gamma_3^{-1} l^{-1} F^{\frac{1}{2}}$,	from (iii$a$)
$[D]$	$\gamma_1^{-\frac{1}{2}}\gamma_5^{-1} l^{-1} F^{\frac{1}{2}}$		$[B]$	$\gamma_2^{-\frac{1}{2}}\gamma_6^{-1} l^{-1} F^{\frac{1}{2}}$,	from (vi)
$[i]$	$\gamma_1^{\frac{1}{2}} l T^{-1} F^{\frac{1}{2}}$	or	$[B]$	$\gamma_1^{-\frac{1}{2}}\gamma_4 l^{-1} F^{\frac{1}{2}}$,	from (iv)
$[\phi]$	$\gamma_1^{-\frac{1}{2}} F^{\frac{1}{2}}$		$[L]$	$\gamma_1^{-1}\gamma_4 T$.	
$[C]$	$\gamma_1 l$				

To show the way in which these are derived, from (i) $q^2 = Kr^2F/\gamma_1$ and hence $[q]^2$ is proportional to $\gamma_1 l^2[F]$, K being independent of the unit system. The units of E follow and from $\mathbf{D} = \gamma_5 K\mathbf{E}$, $[D]$ is proportional to $\gamma_5{}^{-1}[E]$, and so on. From the two expressions for $[H]$ and $[B]$ we see that

$$\gamma_2{}^{-\frac{1}{2}} = \gamma_1^{\frac{1}{2}}\gamma_3{}^{-1}, \quad \text{.....(viii)}$$

$$\gamma_2{}^{-\frac{1}{2}}\gamma_6{}^{-1} = \gamma_1{}^{-\frac{1}{2}}\gamma_4. \quad \text{.....(ix)}$$

It is since these relationships do not involve the units that the gammas can be dimensionless. It is not necessary to introduce any quantity with dimensions other than length, time and force, or what is equivalent, length, time and mass. If it is preferred not to introduce a fractional power all units can be expressed in terms of the units of length, time and charge, by replacing $[F]$ by $\gamma_1{}^{-1} l^{-1} q^2$, in all the expressions in the table.

3. *Systems of units.*

In the first three systems, $l = 1$ cm., $F = 1$ dyne.

A. The electrostatic system.

$\gamma_1 = 1$, $\gamma_2 = \gamma_4 = c$, $\gamma_5 = 1$ are taken as defining the system. From these and equations (viii) and (ix) we find $\gamma_3 = c^2$, $\gamma_6 = 1/c^2$.

B. The electromagnetic system.

$\gamma_2 = 1$, $\gamma_3 = \gamma_4 = c$, $\gamma_6 = 1$ are taken as defining the system. These are consistent with an equation derived from (viii) and (ix) and with (viii) show that $\gamma_1 = c^2$. γ_5 could have any value but is usually taken as $1/c^2$, so as to make div $\mathbf{D} = 4\pi\rho$ in this system as well as in the electrostatic system.

In the two systems A and B and also in the system D which, follows, the velocity of light occurs in the equations for curl $\mathbf{H}$, curl $\mathbf{E}$, only in the form c/c^1, which has measure 1, so if we put $(c^1/c) = 1$, the equations can be checked dimensionally only by assuming that the unit of velocity occurs in the denominator of the right-hand side of each equation.

As an example of the table of dimensions using suffix s for electrostatic and m for electromagnetic values

$$\frac{[R_s]}{[R_m]} = \frac{[\gamma_1^{-1}]_s}{[\gamma_1^{-1}]_m} = c.$$

C. The Gaussian system.

$q, i, E, D, \phi, j, C, R$ are measured in electrostatic units whence $\gamma_1 = 1$, $\gamma_5 = 1$, and m, H, B in electromagnetic units whence $\gamma_3 = \gamma_6 = 1$. From these values and equations (viii) and (ix) we find $\gamma_3 = \gamma_4 = 1$.

D. The rationalized metre-kilogramme-second (or Giorgi) system.

The objects of the system are :

(*a*) to eliminate the velocity of light and 4π from formulae (iii*a*) and (iv). This necessitates $\gamma_3 = c_k/4\pi$, $\gamma_4 = c_k$, $4\pi\gamma_1\gamma_5 = 1$, where c_k^1 is the velocity of light c_k is its measure in the new system.

(*b*) to make the practical units, namely, the joule, coulomb, ampere, farad, volt, ohm and henry become units of the system.

Calling the units of length and force l_k and F_k and for the moment not assuming that they are 1 metre and 1 kilogramme-metre/sec.^{2} respectively, but taking the unit of time to be 1 second throughout, we have

$$l_k F_k = 1 \text{ joule} = 10^7 \text{ ergs} = 10^7 lF,$$

$$\frac{\gamma_1^{\frac{1}{2}}}{c}\frac{l_k}{l}\frac{F_k^{-\frac{1}{2}}}{F^{-\frac{1}{2}}} = \frac{[q_k]}{[q_m]} = \frac{1 \text{ coulomb}}{[q_m]} = 10^{-1}, \quad \text{.....(xi)}$$

$$\frac{\gamma_1}{c^2}\frac{l_k}{l} = \frac{[C_k]}{[C_m]} = \frac{1 \text{ farad}}{[c_m]} = 10^{-9}, \quad \text{.....(xii)}$$

$$\frac{\gamma_1^{-1} l_k^{-1}}{c^{-2} l^{-1}} = \frac{[R_k]}{[R_m]} = \frac{1 \text{ ohm}}{[R_m]} = 10^9, \quad \text{.....(xiii)}$$

$$\frac{\gamma_1^{-\frac{1}{2}} F_k^{\frac{1}{2}}}{c^{-1} F^{\frac{1}{2}}} = \frac{[\phi_k]}{[\phi_m]} = \frac{1 \text{ volt}}{[\phi_m]} = 10^8, \quad \text{.....(xiv)}$$

$$\frac{\gamma_1^{-1}\gamma_4}{c^{-2}.c} = \frac{[L_k]}{[L_m]} = \frac{1 \text{ henry}}{[L_m]} = 10^9. \quad \text{.....(xv)}$$

Using the equation $\gamma_1^{-1}\gamma_4 = 10^9 c^{-1}$, the equations under (*a*) above and equations (viii) and (ix) we find that

$$\gamma_1 = \gamma_4 10^{-9} c = c_k c\ 10^{-9},$$
$$\gamma_2 = \gamma_3^2 \gamma_1^{-1} = c_k 10^9 / c(4\pi)^2,$$
$$\gamma_5 = \gamma_1 \gamma_3^{-1} \gamma_4^{-1} = c \cdot 4\pi / c_k \cdot 10^9.$$

Thus we have found all the gammas without fixing l_k and F_k. To find these we note that if (xii) is satisfied, (xiii) is also satisfied, and when (xii) and (xiv) are satisfied so also are (x) and (xi); we need therefore consider only (xii) and (xiv). From (xii),

$$l_k/l = c^2 \cdot 10^{-9}/\gamma_1 = c/c_k, \quad \text{.....(xvi)}$$

and from (xiv),

$$F_k/F = \gamma_1 10^{16}/c^2 = c_k \cdot 10^7/c. \quad \text{.....(xvii)}$$

Now (xvi) is satisfied automatically because the velocity of light is clT^{-1} and also $c_k l_k T^{-1}$. Hence we can choose l_k, F_k in any way so long as (xvii) is satisfied. Writing the unit of force in terms of units of mass, length and time, this means so long as $M_k l_k / Ml = c_k \cdot 10^7/c$, or combining this with (xvi), $M_k/M = 10^7 l^2/l_k^2$.

A convenient system is to take $l_k = 1$ metre $= 10^2 l$, then

$$M_k = 10^3 M = 10^3 \cdot 1 \text{ gramme} = 1 \text{ kilogramme}.$$

This gives the metre-kilogramme-second system. The fundamental equations (i) to (vii) take the following forms in this system.

$$\frac{{c_k}^2}{10^7} \cdot \frac{q^2}{Kr^2} = F, \quad \frac{10^7}{(4\pi)^2} \frac{m^2}{\mu r^2} = F,$$

$$\operatorname{curl} \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t}, \quad \operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\mathbf{D} = \frac{10^7}{4\pi {c_k}^2} K\mathbf{E}, \quad \mathbf{B} = \frac{4\pi}{10^7} \mu \mathbf{H},$$

$$\operatorname{div} \mathbf{D} = \rho.$$

Some writers prefer to call $10^7/4\pi {c_k}^2$ and $4\pi/10^7$ the dielectric constant and permeability of the ether and denote them by K_0 and μ_0. In this case the first two and last two equations above become

$$q^2/4\pi K K_0 r^2 = F, \quad m^2/4\pi\mu\mu_0 r^2 = F,$$
$$\mathbf{D} = KK_0\mathbf{E}, \quad \mathbf{B} = \mu\mu_0\mathbf{H}.$$

The M.K.S. system has many advantages for the practical man, but the Gaussian system has the advantage that, owing to the velocity of light appearing explicitly and not as an implied unit velocity, it is much easier to check dimensions of formulae in the Gaussian system. H.V.L.

1786. No sensible man would think to improve Newton's *Principia* and Darwin's *Origin of Species* by casting them into blank verse; or Euclid's *Elements* by writing them out in ballad metre:

The King sits in Dunfermline toun,
Drinking the blude-red wine:
"O wha will rear me an equilateral triangle
Upon a given straight line?"

—Sir Arthur Quiller-Couch, *From a Cornish Window* (1928 ed.), p. 34. [Per Rev. A. F. Mackenzie.]

1787. *So with sound waves. Suppose an aircraft is travelling at* 200 *miles an hour a thousand feet from the ground.*

The advance wave it sends out travels at 761 *miles an hour and so is moving* 561 *miles ahead to make an orderly passage for it*—a funnel through the atmosphere.—*News-Chronicle*, 18th June, 1953. [Per Mr. P. Vermes.]

REAL PROJECTIVE GEOMETRY.

By E. J. F. Primrose.

1. In elementary work the distinction between real and complex geometry is generally ignored. For example, it is usual to say that the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents a circle if $a = b$ and $h = 0$. This, as it stands, is true only of complex geometry: if $g^2 + f^2 < ca$, there are no real points whose coordinates satisfy the equation.

Similarly, though it is usually convenient to assume that a line and a conic have always two (possibly coincident) points of intersection, at other times it is of interest to distinguish between the cases where the line does or does not meet the conic. The first is complex geometry and the second is real geometry. The distinction may be put in algebraic form: the first treatment assumes that a quadratic equation has always two (possibly coincident) roots, whereas the second allows only real roots.

In the present paper, we present some well-known theorems of complex projective geometry in forms which are also true in real projective geometry.

The most important device (due to von Staudt) which we use is to replace points of intersection of a line and a conic by the involution of conjugate points determined by the conic on the line. If this involution has double points, these are the points of intersection, but the involution exists whether the line and the conic intersect or not.

Two-dimensional theorems.

2. The following preliminary remarks are necessary.

(i) We use homogeneous coordinates: an equation of the second degree will be said to represent a conic only if there are real values of x, y, z (not all zero) which satisfy the equation.

(ii) In theorems 1 to 6 two involutions I_a, I_b on lines a, b are introduced. It is assumed throughout that C, the common point of a, b is not a double point of either involution.

(iii) We denote by P_a the mate in I_a of any point P on a other than C, and similarly for points on b. The mates of C in I_a, I_b will be denoted by B, A respectively. The line AB will be called c.

(iv) A conic for which I_a, I_b are involutions of conjugate points will be denoted by I_aI_b, and if X is a point of such a conic we speak of the conic XI_aI_b.

(v) In view of (iii), if ABC is taken as triangle of reference, the involutions I_a, I_b may be expressed in the form $zz' = \beta yy'$, $zz' = \alpha xx$.

It follows that a conic I_aI_b is of the form

$$\alpha x^2 + 2\lambda xy + \beta y^2 = z^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where λ is such that the condition in (i) is satisfied, but is otherwise arbitrary.

(vi) The necessary and sufficient condition that two involutions on the same line

$$a\mu\mu' + b(\mu\nu' + \mu'\nu) + c\nu\nu' = 0,$$
$$a'\mu\mu' + b'(\mu\nu' + \mu'\nu) + c'\nu\nu' = 0$$

should be commutative* is easily shown to be

$$ac' + a'c - 2bb' = 0.$$

* R. L. Goodstein, "Commutative Involutions," *Math. Gazette*, vol. XXXI, pp. 224–6.

3. THEOREM 1. *The conics I_aI_b which meet a line u meet it in pairs of points of an involution I.*

If the equation of u is $lx+my+nz=0$, meeting a at J $(0, -n, m)$ and b at K $(-n, 0, l)$, then any two points of u are

$$(-\nu n, -\mu n, \mu m+\nu l), \qquad (-\nu' n, -\mu' n, \mu' m+\nu' l).$$

The condition that both these points should lie on a conic I_aI_b is

$$\mu\mu'(\beta n^2-m^2)-\nu\nu'(\alpha n^2-l^2)=0.$$

The points are therefore pairs of an involution I on u. This involution may be interpreted geometrically: J, K are mates in I, and the points $\begin{pmatrix} JK \\ J_aK_b \end{pmatrix}$, uc are also mates in I.

4. THEOREM 2. *The involution I of theorem 1 is commutative with the involution I_1 of pairs of points on u conjugate with respect to any conic of the system I_aI_b.*

If the conic is $\alpha x^2+2\lambda_1 xy+\beta y^2=z^2$, I_1 is given by

$$\mu\mu'(\beta n^2-m^2)+(\mu\nu'+\mu'\nu)(\lambda_1 n^2-lm)+\nu\nu'(\alpha n^2-l^2)=0,$$

which is commutative with I.

5. THEOREM 3. *The poles of u with respect to the conics I_aI_b lie on a conic.*

The poles may be shown to lie on the conic Θ, whose equation is

$$n(\alpha x^2-\beta y^2)+z(lx-my)=0.$$

Θ corresponds to the "eleven-point conic" of complex projective geometry. We mention four points on Θ and two involutions of conjugate points.

(i) C, J_a, K_b and $\begin{pmatrix} AJ_a \\ BK_b \end{pmatrix}$ lie on Θ.

(ii) The condition that the points

$$(-\nu n, -\mu n, \mu m+\nu l), \quad (-\nu' n, -\mu' n, \mu' m+\nu' l)$$

should be conjugate with respect to Θ is

$$\mu\mu'(\beta n^2-m^2)-\nu\nu'(\alpha n^2-l^2)=0.$$

Hence I is an involution of conjugate points.

(iii) On c the involution I_θ of conjugate points with respect to Θ is given by

$$\alpha xx'-\beta yy'=0.$$

In this involution A, B is a pair of mates, and if JK, J_aK_b meet c in L, L', then L, L' is a pair of mates.

The first three points in (i) and the involutions in (ii) and (iii) correspond to seven of the eleven points specified in complex projective geometry, but the fourth point in (i) is not one of the eleven. The other four of the eleven points do not necessarily have any analogue in our system.

6. DEFINITION. If P and p are pole and polar for a conic Σ, and I is any involution on p commutative with the involution of conjugate points on p, we call (PI) a *self-polar configuration* with respect to Σ. If I has two self-corresponding points Q, R then PQR is a self-polar triangle. It will be observed that a self-polar configuration is a self-dual concept.

7. THEOREM 4. *The conics I_aI_b have a common self-polar configuration.*

For (CI_θ), where I_θ is the involution described in (iii) of theorem 3, satisfies the conditions of the definition.

8. THEOREM 5. *If (XI_a), $(X'I_b)$ are self-polar configurations with respect to a conic, then there is a conic $XX'I_aI_b$.*

If the coordinate system is chosen so that the involutions have the same equations as in previous theorems, the given conic must be of the form

$$\alpha x^2 + \beta y^2 + z^2 + 2fyz + 2gzx + 2hxy = 0$$

and the coordinates of X, X' are easily found. There is then a conic $XX'I_aI_b$ of the form (1), where λ is given by

$$2\lambda(fg - h) = \alpha f^2 + \beta g^2 - \alpha\beta - h^2.$$

Corollary. If two conics Ψ, Σ are such that there is a point X on Ψ and an involution I_a of points conjugate for Ψ which together form a self-polar configuration for Σ, then we can find as many such configurations as we please, and we say that Ψ is *outpolar* to Σ.

Let X' be any other point on Ψ and b its polar with respect to Σ. Let a, b meet at C, and let A be the point on b conjugate to C with respect to Ψ. Let I_b be the involution on b commutative with the involution on b of conjugate points with respect to Σ and in which C, A is a pair of mates. Then $(X'I_b)$ is a self-polar configuration for Σ, and so by theorem 5 there is a conic $XX'I_aI_b$: it is easily verified that Ψ is the only conic $XX'I_a$ for which C, A is a pair of conjugate points. Hence I_b is an involution of conjugate points for Ψ.

9. THEOREM 6. *If a given conic touches two lines a, b, if X, X' are given points, if the involution of conjugate lines through X determines an involution I_a on a and if the involution of conjugate lines through X' determines an involution I_b on b, then there is a conic $XX'I_aI_b$.*

If the coordinate system is chosen so that the involutions have the same equations as in previous theorems, then the given conic must have a line equation of the form

$$Cn^2 + Fmn + Gnl + Hlm = 0,$$

and the coordinates of X, X' are easily found. There is then a conic $XX'I_aI_b$ of the form (1), where λ is given by

$$2\lambda(FG - CH) = \alpha\beta H^2 - \alpha G^2 - \beta F^2 + C^2.$$

Corollary. If two conics Ψ, Σ are such that there is a point X on Ψ and an involution I_a of points conjugate for Ψ such that a touches Σ and the lines joining X to pairs of points of I_a are points of an involution of lines conjugate for Σ, then we can find as many such configurations as we please, and we say that Ψ is *conjugately circumscribed* to Σ.

For, if b is any other tangent to Σ, and I_b the involution on b of points conjugate with respect to Ψ, it is easily shown that there is a point X' such that the lines joining X' to pairs of I_b are pairs of an involution of lines conjugate for Σ. By theorem 6, there is a conic $XX'I_aI_b$, and since Ψ is the only conic XI_aI_b, X' lies on Ψ.

10. THEOREM 7. *The lines (if any) on which the involutions of conjugate points with respect to two given conics are commutative touch a fixed conic.*

If the equations of the two given conics are

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$

$$a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy = 0,$$

the conditions that the line $[l, m, n]$ should satisfy the given condition is

$$\Sigma[(bc' + b'c - 2ff')l^2 + 2(gh' + g'h - af' - a'f)mn] = 0.$$

Three-dimensional theorems.

11. Properties of a twisted cubic may be investigated in a similar way.

A twisted cubic Γ may be defined as a curve which, by suitable choice of coordinate system, may be expressed parametrically in the form

$$x=\theta^3, \quad y=\theta^2, \quad z=\theta, \quad t=1,$$

with the usual convention about an infinite value of θ. The line coordinates (taken in the order p_{01}, p_{02}, p_{03}, p_{23}, p_{31}, p_{12}) of the line joining the two points of Γ with parameters θ_1, θ_2 are

$$(\theta_1^2\theta_2^2,\ \theta_1\theta_2(\theta_1+\theta_2),\quad \theta_1^2+\theta_1\theta_2+\theta_2^2,\quad 1,\quad -(\theta_1+\theta_2),\ \theta_1\theta_2).$$

If we put $\theta_1\theta_2=b/c$, $\theta_1+\theta_2=a/c$, these become $(b^2, ab, a^2-bc, c^2, -ac, bc)$. This suggests that, whether the equation $c\theta^2-a\theta+b=0$ has real roots or not, a line whose coordinates are $(b^2, ab, a^2-bc, c^2, -ac, bc)$ may have some of the properties of a line joining two points of Γ. We shall call such a line a *link* of Γ, and denote it by $<abc>$. If $a^2=4bc$, the link has exactly one point in common with Γ and is called a *tangent*.

It may easily be shown that any quadric which contains Γ is of the form

$$\alpha(y^2-zx)+\beta(z^2-yt)+\gamma(yz-xt)=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

12. THEOREM 8. *There is a unique quadric which contains Γ and any two of its links.*

The condition that a quadric of the form (1) should contain the link $<abc>$ is

$$\alpha b+\beta c+\gamma a=0.$$

If the quadric also contains the link $<a'b'c'>$, then

$$\alpha b'+\beta c'+\gamma a'=0.$$

These two equations determine the ratios of α, β, γ uniquely.

13. THEOREM 9. *There is one and only one link passing through any point not on Γ.*

If the given point is (x, y, z, t), the ratios of a, b, c are determined uniquely in the form

$$a=yz-xt, \quad b=y^2-zx, \quad c=z^2-yt.$$

14. THEOREM 10. *If a non-degenerate quadric contains Γ, the generators of one system are links, and the generators of the other system meet Γ in exactly one point.*

We may, without loss of generality, take the equation of the quadric as $xt-yz=0$. A generator of one system is $x=\mu z$, $y=\mu t$, with line coordinates $(\mu^2, 0, \mu, 1, 0, -\mu)$, showing that the generator is a link of Γ. A generator of the other system is $x=\mu y$, $z=\mu t$, which contains the unique point $(\mu^3, \mu^2, \mu, 1)$ of Γ.

15. DEFINITION. An *involution* of points on Γ is defined by pairs of points whose parameters are connected by a relation of the form

$$\lambda\theta_1\theta_2+\mu(\theta_1+\theta_2)+\nu=0.$$

16. THEOREM 11. *If a non-degenerate quadric Σ contains Γ, the generators of Σ which meet Γ in two points determine an involution on Γ.*

For, if θ_1, θ_2 are the parameters of two points on Γ which lie on a generator of the quadric (1), then

$$\alpha\theta_1\theta_2+\beta+\gamma(\theta_1+\theta_2)=0.$$

17. THEOREM 12. *If m is a link of Γ, there is a unique involution on m which is an involution of conjugate points for every quadric containing Γ.*

If m is $<abc>$, and the coordinates of any two points on m are

$$(\lambda ab,\ b(\lambda c+\mu b),\ \mu ab,\ -\lambda c^2+\mu(a^2-bc)),$$
$$(\lambda' ab,\ b(\lambda' c+\mu' b),\ \mu' ab,\ -\lambda' c^2+\mu'(a^2-bc)),$$

the two points are conjugate with respect to every quadric of the form (1) if and only if

$$2c^2\lambda\lambda'+(2bc-a^2)(\lambda\mu'+\lambda'\mu)+2b^2\mu\mu'=0.$$

18. THEOREM 13. *The quadrics containing Γ and any link m determine an involution on any other line.*

Let X be any point on the given line x, m_0 the link of Γ which passes through X, A the point (whose existence is easily verified) where the plane π containing x and m_0 meets Γ, and M the point where π meets m. Then the section by π of any quadric containing Γ and m is a conic passing through A and M and having a given involution on m_0 as an involution of conjugate points, by theorem 12. By theorem 1, such conics determine an involution I on x, and so the quadrics determine the same involution on x.

Corollary. I is commutative with the involution on x of conjugate points with respect to any quadric of the system.

This follows immediately from theorem 2.

E. J. F. PRIMROSE

1788. My Lord Hallifax [Charles Montagu, 1661–1715] desires you will bespeak him a set of all kinds of mathematical instruments, of the largest sort, in ivory, but adorned as curiously as you please, they being more for furniture than for any use he's likely to put 'em too ; he designs to hang 'em up in his library.—Letter from Vanbrugh to Tonson, quoted by Dobrée, *Essays in biography* (Oxford, 1925), p. 85.

1789. Even ; how odd!

Asquith, having a strong liking for academic attainments, even when they were mathematical, . . . *The Times*, 30 November 1953, obituary notice of E. W. Barnes [Per Professor E. H. Neville].

1790. These . . . errors keep many students to reading speeds several hundred or even thousand per cent. below their maximum.—H. J. Eysenck, *Uses and Abuses of Psychology* (Pelican Book, A 281), p. 166 [Per Mr. F. Sandon].

1791. It was as a bowler that Mr. Foster will be best remembered, and I will try to give an impression of his style and method. Bowling left hand round the wicket with a high delivery—he was 6 feet tall—his action was the personification of grace. A few short steps, a graceful skip, an apparently medium-paced ball through the air, but doubling its speed as it touched the ground, he kept an exceptional length. He swerved a little with the new ball, but the angle of delivery made the ball come into the batsman without swerve.—Sir Pelham Warner, introduction to F. R. Foster, *Cricketing Memories*. [Per Mr. B. M. Brown.]

1792. It's true. The temperature is nearly twice as high under the water. . . . On the boathouse wall, a thermometer registered 25 degrees. McInally pulled out the thermometer he took down [under the water] with him. It showed 47 degrees.—*News-Chronicle*, January 29, 1954. [Per Mr. F. Sandon.]

1793. . . . a mystical intuition that the woe and the joy in the world lie always in perfect balance—like the centrifugal and centripetal impulses which keep the earth spinning because they remain for ever equipollent.—Clifford Bax, Evenings in Albany. [Per Mr. D. Temple Roberts.]

MATHEMATICAL NOTES

2427. *The value of a product whose factors are liable to error.*

THE experiment implied by Mr. Hull in Note 2320 appears to be that of measuring a great many rectangles, all of whose dimensions are such that, when rounded-off to the nearest unit, they lead to the same two measured values for the length and breadth. If it can be assumed that the actual lengths (and breadths) are distributed symmetrically about the measured ones, then Mr. Hull's proof shows that more than half the true areas will be less than the area calculated from the measurements.

A more usual approach to the problem is to assume that there is only one rectangle and that a great many measurements are made of its length and breadth. Then, if the measured values are supposed to be distributed symmetrically about the true ones, a similar proof, in which the rôles of (a, b) and (a_0, b_0) are interchanged, leads to the result that more than half the *calculated* areas will be less than the *true* one.

It seems that, when the errors in measurement are negligible in comparison with the rounding-off errors, Mr. Hull's approach is the right one, but that a reversal of these assumptions leads to the opposite conclusion. Combination of both kinds of error requires an analysis which Mr. Hull's treatment was no doubt intended to avoid. A. E. WILLIAMS and B. W. CONOLLY.

2428. *A remark about boundaries of regions.*

The discussion outlined here does not seem to be one which is usually introduced in textbooks of advanced calculus; and yet it appears to me to give an illuminating generalization of the ideas of maxima and minima of a function of several variables. In particular, from the main result below one can deduce the extreme values of a function $f_1(\theta_1, \dots \theta_n)$ subject to constraints $f_k(\theta_1, \dots \theta_n) = c_k$, $k = 2, 3 \dots m$, among the θ_i.

Consider the m real functions

$$X_k = f_k(\theta_1, \dots \theta_n), \quad k = 1, 2 \dots m, \quad m \leqslant n, \qquad \dots\dots(1)$$

$\theta_1, \dots, \theta_n$ being independent variables, and expand by Taylor's theorem giving

$$\delta X_k = \sum_i \frac{\partial X_k}{\partial \theta_i}\,\delta\theta_i + \frac{1}{2!}\sum_{i,j}\frac{\partial^2 X_k}{\partial\theta_i\,\partial\theta_j}\,\delta\theta_i\,\delta\theta_j + \dots .$$

As $\theta_1, \dots \theta_n$ vary the point $(X_1, \dots, X_m)$ fills some region K (say) of m-space.

Now at a boundary point of K (supposing that a boundary exists) there are increments $\delta X_1, \dots, \delta X_m$ to which correspond (to the first order) no values of the $\delta\theta_i$;* i.e. the equations

$$\delta X_k = \sum_i \frac{\partial X_k}{\partial \theta_i}\,\delta\theta_i, \quad k = 1, 2, \dots m,$$

for $\delta\theta_1, \dots, \delta\theta_n$ are inconsistent for some $\delta X_1, \dots \delta X_m$. Whence the matrix of coefficients on the right hand side, namely

$$\begin{pmatrix} \dfrac{\partial X_1}{\partial \theta_1} & \dots\dots & \dfrac{\partial X_1}{\partial \theta_n} \\ \dots\dots & \dots\dots & \dots\dots \\ \dfrac{\partial X_m}{\partial \theta_1} & \dots\dots & \dfrac{\partial X_m}{\partial \theta_n} \end{pmatrix}$$

must be of rank less than m; i.e. every m-rowed determinant taken from this

* The only increments $\delta X_1, \dots, \delta X_m$ to which there do correspond values of the $\delta\theta_i$ are those for which the point $(X_1 + \delta X_1, \dots, X_m + \delta X_m)$ is a point of the boundary.

matrix must vanish. In general this gives $n-m+1$ independent equations, which together with the m equations (1) gives $n+1$ equations between which one can eliminate $\theta_1, \ldots, \theta_n$ and thus obtain the equation $F(X_1, \ldots X_m)=0$ (say) of a surface, part (or all) of which is the boundary of K.

In particular, solving

$$F(X_1, \ldots X_m)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$X_2=c_2,\ X_3=c_3,\ \ldots,\ X_m=c_m \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

simultaneously gives the points of intersection of the surface (2) with the line (3), and these include the maxima and minima of the function $f_1(\theta_1, \ldots \theta_n)$ subject to the constraints $f_k(\theta_1 \ldots \theta_n)=c_k$, $k=2, 3, \ldots m$.

Example

The given region is $X_1=\sin^2\theta_1\cos\theta_2$, $X_2=\cos^2\theta_1\sin\theta_2$.

For a boundary point it is necessary that

$$\begin{vmatrix} 2\sin\theta_1\cos\theta_1\cos\theta_2 & -\sin^2\theta_1\sin\theta_2 \\ -2\cos\theta_1\sin\theta_1\sin\theta_2 & \cos^2\theta_1\cos\theta_2 \end{vmatrix}=0$$

i.e. $\sin\theta_1\cos\theta_1(\cos^2\theta_1\cos^2\theta_2-\sin^2\theta_1\sin^2\theta_2)$

$$=\sin\theta_1\cos\theta_1\cos(\theta_1-\theta_2)\cos(\theta_1+\theta_2)=0.$$

Whence either $\sin\theta_1=0$ giving $X_1=0$ .....(i)

or $\cos\theta_1=0$ giving $X_2=0$ .....(ii)

or $\cos(\theta_1\pm\theta_2)=0$ giving $\theta_1\pm\theta_2=2r\pi\pm\pi/2$,

and therefore $X_1=\cos^2\theta_2$, $X_2=\sin^2\theta_2$, i.e. $X_1^{2/3}+X_2^{2/3}=1$. .....(iii)

The curve (iii) is the true boundary. HAZEL PERFECT.

2429. *A property of the semi-cubical parabola.*

In Note 2330 Mr. Durell asked for suggestions on the mode of discovery of the property $27n_1n_2n_3n_4=at_1t_2t_3$ connecting the lengths of the four normals and three tangents from a point to the curve $ay^2=x^3$. It might have emerged in the following way.

Taking the point as (ξ, η) we see that the product of the tangents is an expression of the third degree in a, ξ, η, which vanishes with $a\eta^2-\xi^3$; and the product of the normals is of the fourth degree with the same factor. Thus a simple relation between the two might be expected. A similar expectation is true for the products of the *subtangents* and *subnormals*. For these, quite a short analysis yields the results $4(a\eta^2-\xi^3)$ and $\xi(\xi^3-a\eta^2)$; and one would therefore be encouraged to go on to the actual tangents and normals.

A short method is to use $x=3bt^2$, $y=2bt^3$, so that t is the actual gradient, and $4a=27b$. The points of contact of the tangents from (ξ, η) are given by

$$bt^3-\xi t+\eta=0.$$

Transform this to an equation in $\sigma=t^2+1$, namely

$$b^3(\sigma-1)^3-2b\xi(\sigma-1)^2+\xi^2(\sigma-1)-\eta^2=0,$$

giving

$$\sigma_1\sigma_2\sigma_3=\{(\xi+b)^2+\eta^2\}/b^3$$

The square root of this is $\sec\psi_1\sec\psi_2\sec\psi_3$, the multiplier required to convert the product of subtangents to that of tangents, so that (numerically)

$$bt_1t_2t_3=4\sqrt{\{(\xi+b)^2+\eta^2\}}\,.\,(\xi^3-a\eta^2).$$

A similar method for normals gives the secant product as

$$\sqrt{\{(\xi+b)^2+\eta^2\}}/\xi,$$

so that

$$n_1n_2n_3n_4 = \sqrt{\{(\xi+b)^2+\eta^2\}} \cdot (\xi^2 - a\eta^2)$$
$$= \tfrac{1}{4}bt_1t_2t_3$$

or

$$27n_1n_2n_3n_4 = at_1t_2t_3,$$

as required.

A. J. MOAKES.

2430. *Products of lengths of tangents and of normals.*

The answer to Mr. Durell's query (Note 2330, February 1953, p. 58) was given by Laguerre in the *Comptes Rendus* in 1865 (*Œuvres*, II, p. 18). Briefly, the product of the squares of the lengths of the tangents from (p, q) to a curve $f(x, y) = 0$ which has no linear asymptotes is a polynomial $\phi(p, q)$ which is zero if (p, q) is on the curve or is on an isotropic tangent to the curve, but not otherwise, that is, if $f(p, q) = 0$, or if $g(p, q) = 0$, where $g(x, y) = 0$ is the equation of the aggregate of isotropic tangents. The product of the squares of the lengths of the normals from (p, q) is a polynomial $\psi(p, q)$ which has the same factors $f(p, q)$, $g(p, q)$ as $\phi(p, q)$ since there is no difference between an isotropic tangent and an isotropic normal. It can be deduced from the known class of the evolute of a curve that both ϕ and ψ are constant multiples of f^2g, and this establishes the result, apart from identification of a constant factor. The function $g(p, q)$ is a constant multiple of the product of the squares of the distances of (p, q) from the real foci of the curve, and therefore if directions of measurement are ignored the results can be expressed in the simpler form that the product of the lengths of the tangents from (p, q) and the product of the lengths of the normals from (p, q) are two constant multiples of the product of the function $f(p, q)$ by the product of the distances of (p, q) from the real foci. If the curve has linear asymptotes, the product of the squares of the lengths of the tangents from (p, q) is not a polynomial but a rational function whose denominator is the product of the squares of the distances of (p, q) from the asymptotes; the form of the product of the squares of the lengths of the normals is unaffected. The possible factors remain the same if the distance from (p, q) is to a point where the radius from (p, q) cuts the curve at an assigned angle α which is not necessarily a right angle, and the product of the distances from (p, q) to all such points is the product of the lengths of the normals, divided by the appropriate power of $\sin\alpha$.

Laguerre's theorems deserve to be better known. Berzolari includes them in his article in the *Encyklopädie* (III 2 1, p. 398), and Coolidge gives careful proofs (*Algebraic Plane Curves*, p. 176), but two-thirds of a century was a long time for them to wait in comparative obscurity.

If the complete proof of Laguerre's theorems for algebraic curves in general is a serious undertaking, verification in a particular case is always theoretically straightforward. In the case of the semi-cubical parabola $x^3 - 27ay^2 = 0$, (p, q) is on the tangent at the current point $(3au^2, au^3)$ if

$$u(p - 3au^2) - 2(q - au^3) = 0,$$

on the normal at this point if

$$2(p - 3au^2) + u(q - au^3) = 0.$$

Hence the equation for the squares of the lengths of the tangents from (p, q) is the u-eliminant of the pair of equations

$$\{T - (p^2+q^2)\} + 6apu^2 + 2aqu^3 - 9a^2u^4 - a^2u^6 = 0$$
$$2q - pu + au^3 = 0,$$

and the equation for the squares of the lengths of the normals is the u-eliminant of

$$\{N-(p^2+q^2)\}+6apu^2+2aqu^3-9a^2u^4-a^2u^6=0$$

$$2p+qu-6au^2-au^4=0.$$

Eliminating dialytically we have, writing r^2 for p^2+q^2,

$$-a^6\phi(p,q)=\begin{vmatrix} -r^2 & 0 & 6ap & 2aq & -9a^2 & 0 & -a^2 & 0 & 0 \\ 0 & -r^2 & 0 & 6ap & 2aq & -9a^2 & 0 & -a^2 & 0 \\ 0 & 0 & -r^2 & 0 & 6ap & 2aq & -9a^2 & 0 & -a^2 \\ 2q & -p & 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 2q & -p & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 2q & -p & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 2q & -p & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 2q & -p & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 2q & -p & 0 & a \end{vmatrix}$$

$$a^6\psi(p,q)=\begin{vmatrix} -r^2 & 0 & 6ap & 2aq & -9a^2 & 0 & -a^2 & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 6ap & 2aq & -9a^2 & 0 & -a^2 & 0 & 0 \\ 0 & 0 & -r^2 & 0 & 6ap & 2aq & -9a^2 & 0 & -a^2 & 0 \\ 0 & 0 & 0 & -r^2 & 0 & 6ap & 2aq & -9a^2 & 0 & -a^2 \\ 2p & q & -6a & 0 & -a & 0 & 0 & 0 & 0 & 0 \\ 0 & 2p & q & -6a & 0 & -a & 0 & 0 & 0 & 0 \\ 0 & 0 & 2p & q & -6a & 0 & -a & 0 & 0 & 0 \\ 0 & 0 & 0 & 2p & q & -6a & 0 & -a & 0 & 0 \\ 0 & 0 & 0 & 0 & 2p & q & -6a & 0 & -a & 0 \\ 0 & 0 & 0 & 0 & 0 & 2p & q & -6a & 0 & -a \end{vmatrix}$$

Each of the functions $\phi(p,q)$, $\psi(p,q)$ is therefore of degree 8, and this degree must be compounded in each case of the degrees of $f(p,q)$ and $g(p,q)$. Since the degree of $g(p,q)$ is certainly even, and the degree of $f(p,q)$ is 3, only one composition is possible, and therefore the degree of $g(p,q)$ is 2, and $\phi(p,q)$ and $\psi(p,q)$ are both multiples of f^2g. The first determinant includes a term $-a^4p^6$, and the second a term a^6p^8; the isotropic tangents are given by $u=\pm 2i$, the focus is $(-4a, 0)$ and it follows that

$$a^2\phi(p,q)=\psi(p,q)=(p^3-27aq^2)^2\{(p+4a)^2+q^2\}.$$

I can form no conjecture as to what was expected of the Tripos candidate of 1890, nor do I know why Edwards should have regarded the problem as an exercise in differential calculus.

The application of Laguerre's theorems to the conic must be the subject of a separate note. The results are to be found in text-books of the 90's, without emphasis or background.

Laguerre gave no proofs. His theorems were rediscovered by Elling Holst, who gave proofs of them in a modest article in the *Annalen* in 1877 (XI, p. 341), and added a note later in the same volume to say that his enunciations had been completely anticipated. Berzolari couples Holst's name with Laguerre's, but Coolidge gives his own demonstrations and mentions no others. E. H. N.

2431. *On tangents and normals.*

If t_1, t_2 are the lengths of the two tangents and n_1, n_2, n_3 the lengths of the three normals from a point to the curve $ay=x^2$, then it is well known that $4n_1n_2n_3=at_1t_2$. In Note 2330 (*Gazette*, February 1953) Mr. Durell comments

on the similar property for the curve $ay^2 = x^3$, which, with an obvious extension of notation, can be written

$$27n_1n_2n_3n_4 = at_1t_2t_3.$$

The extension to the curve $ay^{2n} = x^{2n+1}$ may be stated thus:

$$(2n+1)^{2n+1} \prod_{r=1}^{2n+2} n_r = a \prod_{r=1}^{2n+1} t_r.$$

The proof is omitted, since the whole theory has been given by Laguerre, and reference may be made to Professor E. H. Neville's note preceding this.

M. JACKSON.

2432. *Expressions for the axes of the Brocard and Lemoine ellipses.*

1. In a triangle ABC, Δ is the area, $2p$ the perimeter, h_a, h_b, h_c the altitudes drawn to the sides $BC = a$, $CA = b$, $AB = c$; Θ the area of the orthogonal pedal triangle of the Lemoine point K; Φ the area of the triangle formed by the medians m_a, m_b, m_c; r, r_a, r_b, r_c the radii of the inscribed and the three escribed circles; ω the Brocard angle; 2α and 2β the axes of the inscribed Brocard ellipse (whose foci are the Brocard points Ω and Ω'); $2x$ and $2y$ the axes of the Lemoine ellipse (whose foci are the Lemoine point K and the centroid G); τ and τ' the Torricelli segments (the invariant segments which join a vertex of ABC to the vertices of the equilateral triangles constructed on the opposite side, internally and externally to the fundamental triangle).

It is known that*

$$4\Delta\sqrt{3} = \tau^2 - \tau'^2; \qquad \sqrt{3} \cdot \tan\omega = (\tau^2 - \tau'^2)/(\tau^2 + \tau'^2).$$

2. From the relations

$$\Delta = pr = (p-a)r_a = (p-b)r_b = (p-c)r_c = \tfrac{1}{2}a\,h_a = \dots,$$

we deduce

$$1/r^2 + 1/r_a^2 + 1/r_b^2 + 1/r_c^2 = (a^2+b^2+c^2)/\Delta^2 = 4(1/h_a^2 + 1/h_b^2 + 1/h_c^2). \quad \dots\dots(\text{I})$$

Since it is known that $4\Delta \cot\omega = a^2+b^2+c^2$, we obtain

$$\begin{aligned}\Delta \tan\omega &= 1/(1/h_a^2 + 1/h_b^2 + 1/h_c^2)\\ &= 4/(1/r^2 + 1/r_a^2 + 1/r_b^2 + 1/r_c^2)\\ &= (\tau^2 - \tau'^2)^2/12(\tau^2+\tau'^2), \quad \dots\dots(\text{II})\end{aligned}$$

and

$$1/h_a^2 + 1/h_b^2 + 1/h_c^2 = \tfrac{1}{4}(1/r^2 + 1/r_a^2 + 1/r_b^2 + 1/r_c^2) = 12(\tau^2+\tau'^2)/(\tau^2-\tau'^2)^2.$$

3. Since†

$$y = \Delta/\sqrt{(m_a^2+m_b^2+m_c^2)} = 2\Delta/\sqrt{\{3(a^2+b^2+c^2)\}};$$

$$y = \tfrac{2}{3}(\Theta\,\Phi)^{1/4}; \qquad y = \sqrt{(\tfrac{1}{3}\Delta\tan\omega)}, \quad \dots\dots(\text{M})$$

then using the results of § 2, we have

$$\begin{aligned}y &= 1/\sqrt{\{3(1/h_a^2 + 1/h_b^2 + 1/h_c^2)\}},\\ &= 2/\sqrt{\{3(1/r^2 + 1/r_a^2 + 1/r_b^2 + 1/r_c^2)\}}\\ &= (\tau^2 - \tau'^2)/6\sqrt{(\tau^2+\tau'^2)}. \quad \dots\dots(\text{N})\end{aligned}$$

* V. G. Cavallaro, *Mathesis*, 1938, p. 174 (Bruxelles).

† V. G. Cavallaro, *Math. Gazette*, 1950, p. 266.

4. If p_a, p_b, p_c are the distances of the Lemoine point K from the sides BC, CA, AB, and if we write $p_a+p_b+p_c=\xi$, then

$$\tan\omega = 2p_a/a = 2p_b/b = 2p_c/c = \xi/p,$$

$$\Delta\tan\omega = h_ap_a = h_bp_b = h_cp_c = \xi r, \quad \ldots\ldots\ldots\ldots(\Sigma)$$

and the last formula of (M), § 3, gives

$$y = \sqrt{(h_ap_a/3)} = \ldots = \sqrt{(\xi r/3)}. \quad \ldots\ldots\ldots\ldots(P)$$

Note that from the equations (Σ) and (II) we have

$$\xi = (1/h_a+1/h_b+1/h_c)/(1/h_a^2+1/h_b^2+1/h_c^2)$$
$$= 4/r(1/r^2+1/r_a^2+1/r_b^2+1/r_c^2).$$

5. We can now prove the very curious results :

$$\alpha = \Delta/\sqrt{(h_a^2+h_b^2+h_c^2)} \qquad \beta = h_ah_bh_c/(h_a^2+h_b^2+h_c^2)$$
$$y = \Delta/\sqrt{(m_a^2+m_b^2+m_c^2)}, \qquad x = m_am_bm_c/(m_a^2+m_b^2+m_c^2) \quad \ldots\ldots\ldots\ldots(Q)$$

From the known formulae

$$2\alpha = 2R\sin\omega, \qquad 2\beta = 4R\sin^2\omega,$$

where R is the radius of the circumcircle of ABC, we deduce

$$2\alpha = abc/\sqrt{(b^2c^2+c^2a^2+a^2b^2)},$$
$$2\beta = 4\Delta abc/(b^2c^2+c^2a^2+a^2b^2).$$

If then, in these formulae, we put $\alpha = 2\Delta/h_a$, ... we obtain the first and second formulae of the group (Q). The third formula of this group has been proved in § 3. It remains to prove the last formula of the group. It was remarked in the article in the *Gazette* to which reference was made in § 3 that the focal axis $2x$ of the Lemoine ellipse can be calculated from the familiar result

$$4x^2 = 4y^2 + GK^2,$$

but we give here a more convenient method. As is known, x is equal to the radius of the circle which passes through the orthogonal projections of the foci G, K on the sides of the fundamental triangle. Thus, if XYZ is the orthogonal pedal triangle of the Lemoine point K, whose area is Θ, we have

$$x = YZ\,.\,ZX\,.\,XY/4\Theta.$$

Now the triangle XYZ, of area Θ, is similar to the triangle, of area Φ, formed by the medians of the triangle ABC and the ratio of similarity is* $\tan\omega$. Thus we have

$$YZ = m_a\tan\omega, \ldots, \quad \Theta = \Phi\tan^2\omega;$$

$$x = (m_am_bm_c/4\Phi)\tan\omega.$$

But it is known that the triangle formed by the medians has the same Brocard angle as the fundamental triangle, and hence

$$\tan\omega = 4\Phi/(m_a^2+m_b^2+m_c^2)$$

and

$$x = m_am_bm_c/(m_a^2+m_b^2+m_c^2),$$

which is what we had to prove.

6. If we wish to express y in terms of the medians m_a, m_b, m_c, we observe that $3\Delta = 4\Phi$ and we can express Φ, by means of Heron's formula, in terms of

* V. G. Cavallaro, *Math. Gazette*, 1950, p. 266.

the medians. Thus if we write $2m = m_a + m_b + m_c$, we have, by the third of the formulae (Q),

$$y = \frac{4}{3}\sqrt{\left\{\frac{m(m-m_a)(m-m_b)(m-m_c)}{m_a^2+m_b^2+m_c^2}\right\}}.$$

VINCENZO G. CAVALLARO.

2433. *Sur l'hyperboloïde des hauteurs d'un tétraèdre.*

1. Soit un tétraèdre quelconque $T \equiv ABCD$ dans lequel $BC=a$, $DA=a'$, $CA=b$, $DB=b'$, $AB=c$, $DC=c'$.

Attribuons aux arêtes BC, CD, DB, AD, AB, AC six nombres quelconques λ_1, λ_2, λ_3, λ_1', λ_2', λ_3' et soient A', B', C', D' les points des plans des faces BCD, CDA, DAB, ABC dont les coordonnées barycentriques dans chaque face sont proportionnelles aux nombres λ attribués aux arêtes de cette face. Les droites AB', AC', AD' rencontrent respectivement les arêtes DC, DB, BC en des points B'', C'', D'' tels que

$$BC''/C''D = \lambda_2'/\lambda_1', \quad CD''/D''B = \lambda_3'/\lambda_2', \quad DB''/B''C = \lambda_1'/\lambda_3'$$

et les droites BB'', CC'', DD'' sont donc concourantes en un point A'''. Il en est ainsi pour les droites analogues correspondant aux trois autres faces du tétraèdre. Par suite, la droite AA''', par exemple, rencontre les droites AA', BB', CC', DD' *qui sont quatre génératrices d'un même hyperboloïde.* Désignant par B_1'', C_1'', D_1'' les intersections de CD, DB, BC avec BA', CA', DA' on a

$$BC_1''/C_1''D = \lambda_1/\lambda_2, \quad CD_1''/D_1''B = \lambda_2/\lambda_3, \quad DB_1''/B_1''C = \lambda_3/\lambda_1.$$

La condition *nécessaire et suffisante* pour que B'' et B_1'' coïncident ainsi que C'' et C_1'', D'' et D_1'' est donc que l'on ait

$$\lambda_1\lambda_1' = \lambda_2\lambda_2' = \lambda_3\lambda_3'. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Dans ce cas, *les quatres droites AA', BB', CC', DD' sont concourantes.**

2. Si l'on désigne aussi par a, a', b, b', c, c' les mesures des dièdres suivant les arêtes BC, DA, CA, DB, AB, DC, par A, B, C, D celles des aires des faces BCD, CDA, DAB, ABC et que l'on pose

$$\left.\begin{array}{lll} \lambda_1 = A\,.\,D\cos a, & \lambda_2 = A\,.\,B\cos c', & \lambda_3 = A\,.\,C\cos b' \\ \lambda_1' = B\,.\,C\cos a', & \lambda_2' = C\,.\,D\cos c, & \lambda_3' = B\,.\,D\cos b \end{array}\right\} \quad \ldots\ldots\ldots(2)$$

les coordonnées barycentriques des points A', B', C', D' correspondant à ces nombres λ dans les faces BCD, CDA, DAB, ABC sont proportionnelles aux quantités

$$(D\cos a, B\cos c', C\cos b'), \quad (A\cos c', C\cos a', D\cos b),$$
$$(B\cos a', D\cos c, A\cos b'), \quad (A\cos a, B\cos b, C\cos c),$$

c'est-à-dire à celles qui représentent les coordonnées barycentriques des pieds des hauteurs du tétraèdre T dans les mêmes faces. Dès lors, les droites AA', BB', CC', DD' sont quatre génératrices d'un même hyperboloïde qui se confondent avec les hauteurs du tétraèdre T. Si les égalités (1) ont lieu, les hauteurs AA', BB', CC', DD' sont concourantes et le tétraèdre T est *orthocentrique*. Dans ce cas, d'après (1), (2) on retrouve, en outre, les égalités fondamentales

$$\cos a \cos a' = \cos b \cos b' = \cos c \cos c',$$

entre les cosinus des dièdres opposés de ce tétraèdre particulier.

V. THÉBAULT.

2434. *A propos de triangles rectangles.*

1. Le problème de M. J. Travers dont M. J. K. Bailey propose une général-

* V. Thébault, *Mathesis*, 1931, 52.

isation (*Gazette*, Note 2227, Septembre 1951) en comporte d'autres.

Si a, b et c désignent les mesures de côtés de l'angle droit et de l'hypoténuse d'un triangle rectangle, on peut considérer les triangles dont tous les côtés sont de la forme $x^2 - k^2$.

Exemple. $(113^2 - 4^2)^2 = (49^2 - 4^2)^2 + (112^2 - 4^2)^2$,

soit $12753^2 = 2385^2 + 12528^2$.

D'autre part, on peut envisager des triangles de la forme $na^2 = b^2 + c^2$, dont voici un exemple,

$$2(11^2 - 1)^2 = (13^2 - 1)^2 + (5^2 - 1)^2.$$

N.B. Voici deux quadrangles en nombres entiers de formes comparables,

$$(3^2 + 13)^2 + 1^2 = (2^2 + 13)^2 + (1^2 + 13)^2$$

et

$$(14^2 + 5)^2 + (5^2 + 5)^2 = (13^2 + 5)^2 + (10^2 + 5)^2.$$

2. Avec le triangle rectangle de côtés 72,96,120 on obtient les égalités

$$72 - 71 = 1^2, \qquad 72 = 11^2 - 7^2,$$
$$96 - 71 = 5^2, \qquad 96 = 11^2 - 5^2,$$
$$120 - 71 = 7^2, \qquad 120 = 11^2 - 1^2.$$

V. THÉBAULT.

2435. *Approximations for* coth θ *and an expansion of* $f(x)$.

It is rather surprising how conveniently $\theta \coth \theta$ can be expressed as a continued fraction. In fact, taking

$$\theta \coth \theta = \left(1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots\right) \Big/ \left(1 + \frac{\theta^2}{3!} + \frac{\theta^4}{5!} + \dots\right)$$

for $|\theta| < \pi$ and performing the usual H.C.F. process it follows quickly that

$$\theta \coth \theta = 1 + \frac{\theta^2/1 \,.\, 3}{1 +} \quad \frac{\theta^2/3 \,.\, 5}{1 +} \quad \frac{\theta^2/5 \,.\, 7}{1 +} \dots . \qquad \dots\dots\dots\dots\dots\dots (1)$$

The successive convergents p_n/q_n to this continued fraction will then give a series of approximations to $\theta \coth \theta$ for small θ. Calculating the first three convergents, we find

$$\frac{p_1}{q_1} = 1 + \frac{\theta^2}{3} = 1 + \frac{4 \,.\, 2}{4 \,.\, 3} \frac{\theta^2}{2!},$$

$$\frac{p_2}{q_2} = \left(1 + \frac{2\theta^2}{5}\right) \Big/ \left(1 + \frac{\theta^2}{15}\right) = \left(1 + \frac{6 \,.\, 4}{6 \,.\, 5} \frac{\theta^2}{2!}\right) \Big/ \left(\frac{6}{6} + \frac{6 \,.\, 4 \,.\, 2}{6 \,.\, 5 \,.\, 4} \frac{\theta^2}{3!}\right)$$

$$\frac{p_3}{q_3} = \left(1 + \frac{3}{7}\theta^2 + \frac{\theta^4}{3 \,.\, 5 \,.\, 7}\right) \Big/ \left(1 + \frac{2\theta^2}{21}\right)$$

$$= \left(1 + \frac{8 \,.\, 6}{8 \,.\, 7} \frac{\theta^2}{2!} + \frac{8 \,.\, 6 \,.\, 4 \,.\, 2}{8 \,.\, 7 \,.\, 6 \,.\, 5} \frac{\theta^4}{4!}\right) \Big/ \left(\frac{8}{8} + \frac{8 \,.\, 6 \,.\, 4}{8 \,.\, 7 \,.\, 6} \frac{\theta^2}{3!}\right).$$

The formation of the successive convergents is now clear ; we take

$$\left.\begin{aligned} p_n &= 1 + \frac{2n(2n-2)}{2n(2n-1)} \frac{\theta^2}{2!} + \frac{2n(2n-2)(2n-4)(2n-6)}{2n(2n-1)(2n-2)(2n-3)} \frac{\theta^4}{4!} + \dots, \\ \theta q_n &= \frac{2n}{2n}\theta + \frac{2n(2n-2)(2n-4)}{2n(2n-1)(2n-2)} \frac{\theta^3}{3!} + \dots . \end{aligned}\right\} \qquad \dots\dots (2)$$

In short, p_n and θq_n are given by

$$\sum^{n} \frac{(2n-k)!}{2n!}\, {}^nC_k(2\theta)^k$$

where, for p_n, k takes the values 0, 2, 4, 6, ... and, for q_n, k takes the values 1, 3, 5, ...

The proof of these expressions may be completed by induction, since it follows easily that

$$p_n = p_{n-1} + a_n p_{n-2},$$
$$q_n = q_{n-1} + a_n q_{n-2},$$

with $a_n = \theta^2/(2n-1)(2n-3)$.

This series of approximations is fairly good since the difference between $\theta \coth \theta$ and p_n/q_n is of order θ^{2n+2}.

Further, we may write the approximation in the form

$$\sinh\theta\left(1+\frac{2n(2n-2)}{2n(2n-1)}\frac{\theta^2}{2!}+\ldots\right) \simeq \cosh\theta\left(\frac{2n}{2n}\theta+\ldots\right).$$

Now, if $D=\partial/\partial t$, $e^{hD}f(t)=f(t+h)$, so that, taking $h=\frac{1}{2}(x-a)$ and putting $t=\frac{1}{2}(x+a)$ after the operation, we have

$$[e^{hD}f(t)]_{t=\frac{1}{2}(x+a)} = [f(t+h)]_{t=\frac{1}{2}(x+a)} = f(x),$$

and

$$[e^{-hD}f(t)]_{t=\frac{1}{2}(x+a)} = f(a).$$

Hence, writing $\theta = hD$ and operating on $f(t)$, we obtain an expansion for $f(x)$, namely,

$$\{f(x)-f(a)\}+\frac{2n(2n-2)}{2n(2n-1)}\frac{h^2}{2!}\{f''(x)-f''(a)\}+\ldots$$
$$\simeq \frac{2n}{2n}\{f'(x)+f'(a)\}h+\frac{2n(2n-2)(2n-4)}{2n(2n-1)(2n-2)}\frac{h^3}{3!}\{f'''(x)+f'''(a)\}$$

or $$f(x)-f(a) \simeq \sum_{k=1}^{n}\frac{(2n-k)!}{(2n)!}\,{}^nC_k\{f^k(a)-(-1)^k f^k(x)\}(x-a)^k, \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

The expression on the right is of the Euler-Maclaurin type but contains only n terms. It is actually a particular case deducible from Darboux' expansion* and can alternatively be obtained by successive integration by parts of

$$\int_a^x f^{2n+1}(t)(x-t)^n(t-a)^n\,dt$$

and this gives the remainder R_n after n terms on the right of (3) as†

$$R_n = \frac{(-1)^n(n!)^2(x-a)^{2n+1}}{(2n)!\ (2n+1)!}f^{2n+1}(\xi)$$

where ξ lies between x and a.

The approximation (3) seems to be a particularly good one and has a wide variety of applications. Taking for simplicity the case $n=2$,

$$f(x)-f(a) \simeq \tfrac{1}{2}(x-a)\{f'(x)+f'(a)\}-\tfrac{1}{12}(x-a)^2\{f''(x)-f''(a)\},$$

* Whittaker and Watson, *Modern analysis*, p. 125.

† Hummel and Seebeck, *American Mathematical Monthly*, April 1949 (56), 4, p. 243.

and, putting $f(x)=\sqrt{x}$ and $h=x-a$, we have, after a little reduction, the approximation

$$\left(\frac{x}{a}\right)^3 \simeq \frac{48x^2-12xh-h^2}{48a^2+12ah-h^2}.$$

Again, taking $n=3$ and writing

$$f(x)-f(a)=\int_a^x g(t)\,dt,$$

we get an approximation for the definite integral

$$\int_a^x g(t)\,dt \simeq \tfrac{1}{2}h\{g(x)+g(a)\}-\tfrac{1}{10}h^2\{g'(x)-g'(a)\}+\tfrac{1}{120}h^3\{g''(x)+g''(a)\}.$$

A further approximation is suggested by (3), for if a suitable contour C is chosen,

$$f^k(x) = \frac{k!}{2\pi i}\int_c \frac{f(t)\,dt}{(t-x)^{k+1}}$$

whence (3) suggests

$$\frac{1}{t-x}-\frac{1}{t-a}\simeq\sum_{k=1}^{n}\frac{(2n-k)!n!}{(2n)!(n-k)!}\left\{\frac{1}{(t-a)^{k+1}}-\frac{(-1)^k}{(t-x)^{k+1}}\right\}(x-a)^k.$$

Writing $$t-a=re^{\phi},\ t-x=re^{-\phi},\ x-a=2r\sinh\phi,$$

we have a further approximation

$$\sinh^2\phi-\frac{2n}{2n}\cosh 2\phi\sinh^2\phi+\frac{2n(2n-2)}{2n(2n-1)}\sinh 3\phi\sinh^3\phi$$
$$-\frac{2n(2n-2)(2n-4)}{2n(2n-1)(2n-2)}\cosh 4\phi\sinh^4\phi+\ldots=O(\phi^{2n+2}),$$

where the series on the left terminates after $(n+1)$ terms.

W. R. ANDRESS.

2436. *Some elliptic integrals.*

1. *Introduction.*

A problem in aerodynamics involves the tabulation of the integrals

$$F_0(r)=\int_0^{\pi}\frac{d\phi}{\sqrt{(\cos^2\phi+2r\cos\theta\cos\phi+r^2)}}, \quad \ldots\ldots\ldots\ldots(1)$$

$$F_1(r)=\int_0^{\pi}\frac{\cos\phi\,d\phi}{\sqrt{(\cos^2\phi+2r\cos\theta\cos\phi+r^2)}}, \quad \ldots\ldots\ldots\ldots(2)$$

for the whole range of r from 0 to ∞.

The calculation of an integral can usually be performed in many ways. Sometimes it can be expressed in terms of mathematical functions already tabulated in convenient form; often, a differential equation can be found whose solution by numerical methods gives the required results. In many cases numerical quadrature is the only practicable process.

In this connection the functions $F_0(r)$ and $F_1(r)$ are of some interest. The former is clearly an elliptic integral, but its importance lies in the fact that it can be expressed as a complete elliptic integral, for which convenient and adequate tables already exist. The latter cannot be so expressed, but its tabulation is materially assisted by the use of the former.

I give below the derivation of the formulae for $F_0(r)$ and a similar integral, and also the recommended method of calculation of $F_1(r)$. Some comments are made on allied integrals.

2. *Evaluation of* $F_0(r)$.

We note first that the quadratic cannot vanish for any non-zero r, and that r can be taken to be positive without loss of generality.

Write $t = \tan \phi/2$, then

$$F_0 = \int_0^\infty \frac{2dt}{\sqrt{[t^4(r^2 - 2r\cos\theta + 1) + 2t^2(r^2 - 1) + r^2 + 2r\cos\theta + 1]}}.$$

Let

$$t^4(r^2 - 2r\cos\theta + 1) + 2t^2(r^2 - 1) + r^2 + 2r\cos\theta + 1$$
$$= (r^2 - 2r\cos\theta + 1)(t^2 - 2\alpha t + \alpha^2 + \beta^2)(t^2 + 2\alpha t + \alpha^2 + \beta^2).$$

Then

$$\alpha^2 = \tfrac{1}{2}\left[\frac{r^2 + 2r\cos\theta + 1}{r^2 - 2r\cos\theta + 1}\right]^{1/2} - \tfrac{1}{2}\frac{r^2 - 1}{r^2 - 2r\cos\theta + 1}$$

$$\beta^2 = \tfrac{1}{2}\left[\frac{r^2 + 2r\cos\theta + 1}{r^2 - 2r\cos\theta + 1}\right]^{1/2} + \tfrac{1}{2}\frac{r^2 - 1}{r^2 - 2r\cos\theta + 1}.$$

Following the usual procedure, write

$$t^2 - 2\alpha t + \alpha^2 + \beta^2 = A(t - \gamma)^2 + B(t - \delta)^2,$$
$$t^2 + 2\alpha t + \alpha^2 + \beta^2 = C(t - \gamma)^2 + D(t - \delta)^2.$$

Then

$$\gamma = -\delta = \sqrt{(\alpha^2 + \beta^2)},$$
$$A = D = \tfrac{1}{2}(1 + \alpha/\gamma),$$
$$B = C = \tfrac{1}{2}(1 - \alpha/\gamma),$$

Put $\dfrac{t - \gamma}{t + \gamma} = u$, then $\dfrac{dt}{du} = \dfrac{(t + \gamma)^2}{2\gamma}$, and

$$F_0 = [(r^2 + 2r\cos\theta + 1)(r^2 - 2r\cos\theta + 1)]^{-\frac{1}{4}} \int_{-1}^{1} \frac{du}{\sqrt{\{(Au^2 + 1 - A)(A + \overline{1 - A}u^2)\}}}.$$

If $$I = \int_0^1 \frac{du}{\sqrt{\{(Au^2 + 1 - A)(A + \overline{1 - A}u^2)\}}}$$

and $$u\sqrt{\{A/(1 - A)\}} = \tan x,$$

then

$$I = \int_0 \frac{dx}{\sqrt{\{(A^2\cos^2 x + (1 - A)^2\sin^2 x\}}}, \quad \text{where } \tan x_0 = \sqrt{\{A/(1 - A)\}}$$

$$= \int_{x_0}^{\frac{\pi}{2}} \frac{dy}{\sqrt{\{(1 - A)^2\sin^2 y + A^2\cos^2 y\}}}, \quad \text{where } \tan x \tan y = A/(1 - A).$$

Hence, $$I = \tfrac{1}{2}\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{\{(1 - A)^2\sin^2 x + A^2\cos^2 x\}}}$$

$$= \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{\{\cos^2 x + 4A(1 - A)\sin^2 x\}}}$$

(by a well-known result, due to Gauss)

that is,
$$I = \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{\{1-(2A-1)^2\sin^2 x\}}} = K(c),$$

where $c=(2A-1)^2$, and K is a complete elliptic integral of the first kind.

Thus
$$F_0 = 2\{(r^2+2r\cos\theta+1)(r^2-2r\cos\theta+1)\}^{-\frac{1}{4}} K(c)$$

where
$$c=\frac{\alpha^2}{\alpha^2+\beta^2}=\tfrac{1}{2}-\tfrac{1}{2}\frac{r^2-1}{\sqrt{\{(r^2+2r\cos\theta+1)(r^2-2r\cos\theta+1)\}}}. \quad \text{.....(3)}$$

3. *A similar integral.*

We can apply the same treatment to the function
$$G_0(r)=\int_0^{\pi}\frac{d\phi}{\sqrt{(r^2-2r\cosh\theta\cos\phi+\cos^2\phi)}}, \quad \text{.....(4)}$$

where now $r>e^{\theta}$, to prevent the quadratic ever becoming negative. With the same transformation as before
$$G_0(r)=\int_0^{\infty}\frac{2dt}{\sqrt{\{t^4(r^2-2r\cosh\theta+1)+2t^2(r^2-1)+r^2+2r\cosh\theta+1\}}}.$$

Let
$$t^4(r^2-2r\cosh\theta+1)+2t^2(r^2-1)+r^2+2r\cosh\theta+1 = (r^2-2r\cosh\theta+1)(t^2+A^2)(t^2+B^2).$$

Then
$$A^2=\frac{2r\sinh\theta+r^2-1}{r^2-2r\cosh\theta+1}, \qquad B^2=\frac{2r\sinh\theta-(r^2-1)}{r^2-2r\cosh\theta+1},$$

and are positive if $r>e^{\theta}$.

But
$$\int_0^{\infty}\frac{dt}{\sqrt{\{(t^2+A^2)(t^2+B^2)\}}}=\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{\{A^2\sin^2 x+B^2\cos^2 x\}}} \quad \text{if } t=A\tan x$$
$$=\frac{2}{A+B}K\left[\left(\frac{A-B}{A+B}\right)^2\right].$$

Therefore
$$G_0(r)=\frac{4}{\sqrt{\{2r\sinh\theta+r^2-1\}}+\sqrt{\{2r\sinh\theta-r^2+1\}}}K(p)$$
where
$$p=\left(\frac{\sqrt{\{2r\sinh\theta+r^2-1\}}-\sqrt{\{2r\sinh\theta-r^2+1\}}}{\sqrt{\{2r\sinh\theta+r^2-1\}}+\sqrt{\{2r\sinh\theta-r^2+1\}}}\right)^2 \Bigg\}. \quad \text{.....(5)}$$

4. *Evaluation of F_1.*

For large r the function F_1 can be evaluated fairly easily by expanding the integrand as a power series in $\cos\phi$, and integrating term by term. The result is a power series in r^{-1}, whose coefficients are Legendre Polynomials in $\cos\phi$, again adequately tabulated.

For smaller values of r we can obtain F_1 from the expression:
$$(1+r^2)\frac{dF_1}{dr}=-F_0\cos\theta-2r\cos\theta\frac{dF_0}{dr},$$

where F_0 is given by equation (3), and

$$\frac{dF_0}{dr} = (-2/r)(r^2+1)\{(r^2+2r\cos\theta+1)(r^2-2r\cos\theta+1)\}^{-\frac{1}{2}}E(c)$$
$$+\left\{\frac{2(r^2+1)(1-c)}{r} - \frac{2r(1-2\cos^2\theta+r^2)}{\{(r^2+2r\cos\theta+1)(r^2-2r\cos\theta+1)\}^{\frac{1}{2}}}\right\}$$
$$\{(r^2+2r\cos\theta+1)(r^2-2r\cos\theta+1)\}^{-\frac{1}{2}}K(c). \quad \ldots\ldots\ldots\ldots(6)$$

Performing the necessary substitutions and reduction, we get

$$\frac{dF_1}{dr} = 2\cos\theta\{(r^2+2r\cos\theta+1)(r^2-2r\cos\theta+1)\}^{-\frac{1}{2}}\{2E(c)-K(c)\}$$
$$= Q(r), \quad \text{say}, \quad \ldots\ldots\ldots\ldots(7)$$

a simple type of differential equation equivalent to the straightforward quadrature

$$F_1(r) = F_1(r_0) + \int_{r_0}^{r} Q(x)\,dx, \quad \ldots\ldots\ldots\ldots(8)$$

where r_0 is a large value of r at which F_1 is already known.

Finally, the accuracy of the whole process can be established by comparing, for a small r, the result obtained from (8) and that given by an ascending series in r, or by direct numerical quadrature of (2).

5. Allied integrals.

It can be shown that the more general integrals of the type

$$\int_0^{\pi} \frac{\cos^n\phi\,d\phi}{(r^2+2r\cos\theta\cos\phi+\cos^2\phi)^{\pm m+\frac{1}{2}}}$$

can all be expressed in terms of F_0, dF_0/dr, dF_1/dr and F_1, of which the first three are given explicitly in equations (3), (6) and (7) respectively.

Finally, the integrals obtained from (1), (2) and (4) by replacing $\cos\phi$ by $\sin\phi$ cannot apparently be evaluated in terms of complete elliptic integrals, since they correspond to the original integrals taken between limits 0 and $\frac{1}{2}\pi$, instead of between 0 and π. They can clearly be written as incomplete elliptic integrals, a less satisfactory result from a computational viewpoint.

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A. R. C.

2437. *On functions of the roots of a quartic.*

Mathematics being the art of abbreviation, a roundabout proof is seldom a good proof, but sometimes even in mathematics there is both pleasure and profit to be found along the by-roads. If we look for the condition that the roots of the quartic

$$x^4 - px^3 + qx^2 - rx + s = 0$$

may be separated into pairs with a given product t we have only to identify the product

$$(x^2 - \lambda x + t)(x^2 - \mu x + t)$$

with the given quartic, giving the necessary and sufficient conditions

$$r = pt, \quad s = t^2.$$

Suppose, however, that we ignore the short cut and examine the question

from first principles, what do we find on the way? Calling the roots of the quartic $\alpha, \beta, \gamma, \delta$ the condition that the roots separate into pairs with product t is expressed by the *disjunction*

$$(\alpha\beta = t \text{ and } \gamma\delta = t) \quad \text{or} \quad (\alpha\gamma = t \text{ and } \beta\delta = t) \quad \text{or} \quad (\alpha\delta = t \text{ and } \beta\gamma = t)$$

which (by a simple exercise in symbolic logic) is equivalent to the *conjunction* of

$$(\alpha\beta = t \quad \text{or} \quad \alpha\gamma = t \quad \text{or} \quad \beta\gamma = t)$$

and the seven other like disjunctions of three equations which may be chosen from the six equations obtained by equating the product of a pair of roots to t; and this in turn is equivalent to the eight simultaneous equations

(1) $(t-\alpha\beta)(t-\alpha\gamma)(t-\beta\gamma)=0$,
(2) $(t-\alpha\beta)(t-\beta\delta)(t-\alpha\delta)=0$,
(3) $(t-\gamma\delta)(t-\alpha\gamma)(t-\alpha\delta)=0$,
(4) $(t-\gamma\delta)(t-\beta\delta)(t-\beta\gamma)=0$,
(5) $(t-\alpha\beta)(t-\alpha\gamma)(t-\alpha\delta)=0$,
(6) $(t-\alpha\beta)(t-\beta\delta)(t-\beta\gamma)=0$,
(7) $(t-\gamma\delta)(t-\alpha\gamma)(t-\beta\gamma)=0$,
(8) $(t-\gamma\delta)(t-\beta\delta)(t-\alpha\delta)=0$.

Under the transformation of interchanging α and β, four of these equations are invariant and the remainder fall into two linked pairs according to the following substitution group

$$S_1 = \begin{pmatrix} 1&2&3&4&5&6&7&8 \\ 1&2&4&3&6&5&7&8 \end{pmatrix} = \binom{1}{1}\binom{2}{2}\binom{3\ 4}{4\ 3}\binom{5\ 6}{6\ 5}\binom{7}{7}\binom{8}{8}.$$

Similarly the interchanges (α, γ), (α, δ), (β, γ), (β, δ), (γ, δ) are represented by the groups

$$S_2 = \binom{1}{1}\binom{2\ 4}{4\ 2}\binom{3}{3}\binom{5\ 7}{7\ 5}\binom{6}{6}\binom{8}{8}, \quad S_3 = \binom{1\ 4}{4\ 1}\binom{2}{2}\binom{3}{3}\binom{5\ 8}{8\ 5}\binom{6}{6}\binom{7}{7},$$

$$S_4 = \binom{1}{1}\binom{2\ 3}{3\ 2}\binom{4}{4}\binom{5}{5}\binom{6\ 7}{7\ 6}\binom{8}{8}, \quad S_5 = \binom{1\ 3}{3\ 1}\binom{2}{2}\binom{4}{4}\binom{5}{5}\binom{6\ 8}{8\ 6}\binom{7}{7},$$

$$S_6 = \binom{1\ 2}{2\ 1}\binom{3}{3}\binom{4}{4}\binom{5}{5}\binom{6}{6}\binom{7\ 8}{8\ 7}.$$

Under each transformation the conjunction of the first four equations is invariant and is therefore a symmetric function of the roots of the given quartic, and so too the conjunction of the other four equations is a symmetric function of the roots.

Equation (1) is equivalent to

$$t^3 - t^2(\alpha\beta + \beta\gamma + \gamma\alpha) + t(\alpha^2\beta\gamma + \beta^2\gamma\alpha + \gamma^2\alpha\beta) - \alpha^2\beta^2\gamma^2 = 0,$$

which, writing

$$\alpha\beta + \beta\gamma + \gamma\alpha = q - \delta(p - \delta),$$
$$\alpha^2\beta\gamma + \beta^2\gamma\alpha + \gamma^2\alpha\beta = (s/\delta)(p - \delta), \ \alpha^2\beta^2\gamma^2 = s^2/\delta^2,$$

becomes

$$\delta^4 - p\delta^3 + (q - t + s/t)\delta^2 - (ps/t)\delta + s^2/t^2 = 0,$$

and therefore the conjunction of the equations (1), (2), (3) and (4) says that $\alpha, \beta, \gamma, \delta$ are the roots of the quartic

$$x^4 - px^3 + (q - t + s/t)x^2 - (ps/t)x + s^2/t^2 = 0;$$

thus this quartic must be identical with

$$x^4 - px^3 + qx^2 - rx + s = 0$$

which is true if, and only if, $s = t^2$ and $r = pt$.

Turning next to the second set of four equations, we observe that equation (5) is equivalent to

$$\alpha^4 - p\alpha^3 + (q + t - s/t)\alpha^2 - pt\alpha + t^2 = 0,$$

so that the conjunction of the equations (5), (6), (7), and (8) says that α, β, γ, δ are the roots of the quartic

$$x^4 - px^3 + (q + t - s/t)x^2 - ptx + t^2 = 0,$$

which is true if, and only if, $s = t^2$ and $r = pt$.

Thus we have shown, not just that the necessary and sufficient conditions for the given quartic to have two pairs of roots with product t are $s = t^2$ and $r = pt$, but also *if the six products of two out of four numbers are divided into three pairs of products, each pair containing all four of the given numbers, (as factors of the terms), then the eight cubics which may be formed with one root taken from each pair separate into two classes of four cubics such that the cubics in either class vanish simultaneously if, and only if, the cubics in the other class vanish simultaneously.*

R. L. GOODSTEIN.

2438. *Fermat Numbers.*

Dr. A. E. Western has proved that any factor f of F_n must satisfy

$$f \equiv 1 \pmod{2^{n+2}}.$$

The factors of F_5 and F_6 being known, any search for factors of F_n can be confined to primes $p \equiv 1 \pmod{512}$.

Dr. Western and I have independently tabulated such primes from 10^6 to the end of Lehmer's table, that is, a little beyond 10^7, and collated our tables. I find that that none of these primes is a factor of any F_n except for the known factor of F_9.

The process I used for each p was :

(i) find $2^{32} \pmod p$, least positive and negative values ;

(ii) compare the even value with $2^{31} \pmod p$;

(iii) continue squaring (mod p) as far as necessary, taking in each case both positive and negative residues.

W. BARRETT.

2439. *On Note* 2383 : *Pythagorean triangles.*

$$(p+2)^2 + (q+2)^2 - 4 = (p+q+2)^2 - 2pq.$$

E. H. N.

2440. *On the length of the arc of a curve.*

Consider the curve

$$y = ax^{2+\lambda}, \qquad (a>0,\ \lambda>0). \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Let s be the length of its arc, measured from any fixed point to the point (x, y). The derivative dy/dx is easily seen to be of the form by^{μ}, $(b>0,\ \mu>\frac{1}{2})$ and hence the arc s is given by an integral of the form

$$s = \int \sqrt{\left(1 + \frac{c}{y^{1+\nu}}\right)}\, dy \quad (c>0,\ \nu>0).$$

Consequently,

$$\begin{aligned} s + \text{const.} &= \int \left(1 + \frac{c}{2y^{1+\nu}} - \frac{c^2}{8y^{2+2\nu}} \ldots\right) dy \\ &= y + O(y^{-\nu}). \end{aligned}$$

It follows that $s - y \to$ const. when $y \to \infty$.

Corollary. If s is the length of the arc of any curve that rises more rapidly than the curve (i) when x is large, for example, the graph of any polynomial of the third or higher degree, then $s - y \to$ const. when $y \to \infty$. F. BOWMAN.

2441. *On Note* 2117 : *a geometrical construction for the harmonic mean.*

It may perhaps be of interest to observe that the result obtained for a triangle right-angled at A, namely,

$$\Delta = xy,$$

where the incircle touches BC at D and $BD = x$, $DC = y$, leads to a construction for the harmonic mean of two given lengths. For, if the altitude AH of the triangle is of length h,

$$\Delta = \tfrac{1}{2}h(x + y) = xy,$$

so that

$$h = 2xy/(x + y),$$

and h is therefore the harmonic mean of x and y.

Thus if x and y are given, their harmonic mean may be constructed geometrically as follows.

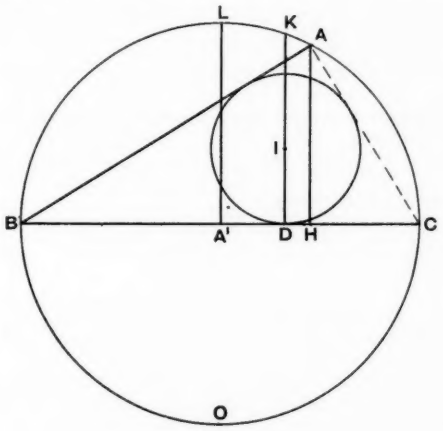

FIG.

Along one line lay off $BD = x$, $DC = y$. Draw the circle on BC as diameter. Let the perpendicular bisector of BC cut this circle in O. With centre O and radius OB draw a circle to cut the perpendicular through D to BC in I. Draw the circle with centre I and radius ID. Let the tangent from B to this circle cut the circle OBC in A. Then the perpendicular AH from A to BC is the required harmonic mean. As a check, the tangent from C to the circle with centre I should also pass through A.

Although this construction is of course much more complicated than what I believe is the standard construction for a harmonic mean (as for example in Hall and Stevens, *School geometry*, p. 325), it may nevertheless be useful as an exercise in accurate geometrical drawing, involving as it does a variety of

elementary constructions for perpendiculars, tangents, etc. The proof of the correctness of the construction is another of the problems which may be derived from this figure.

The diagram shows clearly the relative magnitudes of the arithmetic, geometric and harmonic means, these being respectively $A'L$, DK and HA.

A. J. HOWIE.

2442. *Note on Foucault's Pendulum.*

The vector equation which governs the horizontal motion of the bob of a Foucault's pendulum, suspended from a point near the earth's surface in latitude λ, is given by

$$\frac{\partial^2 \boldsymbol{\rho}}{\partial t^2} + 2\omega \sin \lambda (\mathbf{k} \wedge \frac{\partial \boldsymbol{\rho}}{\partial t}) + n^2 \boldsymbol{\rho} = 0, \quad \ldots\ldots(1)$$

where $\boldsymbol{\rho}$ represents the horizontal displacement of the bob from the vertical through the point of suspension, $\frac{\partial \boldsymbol{\rho}}{\partial t}$ the rate of change of $\boldsymbol{\rho}$ with respect to a set of axes fixed relative to the earth, ω is the earth's angular velocity, $\mathbf{k}$ is a unit vector in the direction of the upward vertical and $n^2 = g/l$, l being the length of the pendulum. The derivation of the above equation is to be found in Milne, *Vectorial Mechanics*, p. 248, § 287.

To obtain the nature of the motion represented by the above equation, Milne seeks a solution of the type

$$\frac{\partial \boldsymbol{\rho}}{\partial t} = \varpi (\mathbf{k} \wedge \boldsymbol{\rho}), \quad \ldots\ldots(2)$$

where ϖ is a constant to be determined. He finds that two solutions are possible, the corresponding values of ϖ being the roots of the quadratic

$$\varpi^2 + 2\varpi\omega \sin \lambda - n^2 = 0. \quad \ldots\ldots(3)$$

Neglecting ω^2, these are given by

$$\varpi_1, \varpi_2 = -\omega \sin \lambda \pm n. \quad \ldots\ldots(4)$$

The path of the bob is therefore given by

$$\boldsymbol{\rho} = \boldsymbol{\rho}_1 + \boldsymbol{\rho}_2, \quad \ldots\ldots(5)$$

where $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ are vectors of constant modulus rotating with angular velocities ϖ_1 and ϖ_2 respectively. From this result the standard interpretation of the motion is deduced.

If the above method is not used then the usual method of treatment is to integrate the equation, the integration being carried out by first replacing $\boldsymbol{\rho}$ by the complex variable $(x + iy)$ and then seeking a solution of the transformed equation in the standard way (see, for instance, Jeffreys and Jeffreys, *Methods of Mathematical Physics*, 2nd ed., p. 113, Ex. (11)).

The nature of the motion, however, can be deduced very simply by a purely vector treatment, which does not involve the integration of equation (1). To show this, consider a particle moving in a plane under the action of a central force of attraction $n^2\mathbf{r}$. Then referred to a set of axes fixed in space and having their origin at the centre of attraction O, the equation of motion of the particle is

$$\frac{d^2\mathbf{r}}{dt^2} + n^2\mathbf{r} = 0, \quad \ldots\ldots(6)$$

and the path of the particle is in general an ellipse having its centre at O and described with a periodic time $2\pi/n$.

If the same motion is referred to a set of axes rotating about the perpendicular to the plane through O with a uniform angular velocity $\omega(\omega < n)$ then, since the acceleration of the particle referred to these axes is given by

$$\frac{\partial^2 \mathbf{r}}{\partial t^2} + 2\omega(\mathbf{k} \wedge \frac{\partial \mathbf{r}}{\partial t}) - \omega^2 \mathbf{r}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

the corresponding equation of motion would be

$$\frac{\partial^2 \mathbf{r}}{\partial t^2} + 2\omega(\mathbf{k} \wedge \frac{\partial \mathbf{r}}{\partial t}) + (n^2 - \omega^2)\mathbf{r} = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

where $\mathbf{k}$ is a unit vector perpendicular to the plane of the motion. Consequently we have the general result that the solution of the differential equation of the type

$$\frac{\partial^2 \mathbf{r}}{\partial t^2} + 2\omega(\mathbf{k} \wedge \frac{\partial \mathbf{r}}{\partial t}) + \mu^2 \mathbf{r} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

referred to a set of axes rotating with an angular velocity ω about an axis perpendicular to the plane of the motion, corresponds to motion in a central ellipse which, when referred to a set of fixed axes, is defined by the equation

$$\frac{d^2 \mathbf{r}}{dt^2} + (\mu^2 + \omega^2)\mathbf{r} = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

If we now apply the above interpretation to equation (1) we see that the path of the bob will be the central ellipse which, when the motion is referred to axes whose directions in space are fixed, corresponds to the solution of the equation

$$\frac{d^2 \boldsymbol{\rho}}{dt^2} + (n^2 + \omega^2 \sin^2 \lambda)\boldsymbol{\rho} = 0\,. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

To an observer moving with the earth the path of the bob thus appears to be an ellipse, described with a periodic time

$$T = 2\pi/(n^2 + \omega^2 \sin^2 \lambda)^{\frac{1}{2}} \simeq 2\pi/n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

if ω^2 is neglected, which rotates about the vertical with an angular velocity $-\omega \sin \lambda$.

C. E. EASTHOPE.

2443. *The Idea of Dynamic Patterns in Geometry.*

The purpose of this article is to present an idea which is being successfully put into practice in a number of schools by teachers who were formerly our students. It is related to geometry, but it is based upon the understanding of what constitutes the foundations of geometrical thinking.

Every teacher of mathematics is aware that his work involves a two-sided process. On the one hand, there are his pupils, and on the other his subject. The latter has to a certain extent been mastered in his college studies, and he sees mathematics as being a set of concepts and relationships linked together logically. It seems to him that any presentation of his subject must to some degree involve these notions and this way of thinking, and when he observes that his pupils do not use these tools as he does, he attempts to alter either the order of succession of the various topics, or his method. The whole problem of teaching mathematics seems to revolve round this difficulty of relating children's spontaneous thought to mathematical rigour and abstractness.

Our solution to this difficulty has been, not to water down the syllabus, but to ask ourselves how one's experience becomes mathematical. It is obvious that those who become mathematicians were at one time children, and that they had to construct a mathematical type of mental activity to which they

later adhered. We shall be concerned here only with the question of geometry at the school level, and more particularly with the relation of geometry to spatial experience.

In our view, the whole of school geometry is within the grasp of every adolescent capable of concentration upon his perceptual and active experience, provided that this is presented to him in an organised form which yields geometrical facts. This is the basis of the method of dynamic patterns.

As children, we learn in our games to organise perception for the purpose of action, and to organise perception and action with a view to more complicated action and perception. The perception that concerns us here is that of geometrical facts. These, for the adult, are contained in geometrical situations, but from the point of view of the teacher they are hidden in the situation and have to be extracted, *abstracted*. Our idea is to animate any given figure so as to form a pattern containing an infinite number of figures, but to do so in such a way that only the fact that we want to abstract shall be singled out. If we begin with a circle, for instance, we can arrive at a variety of situations, according as we make the radius vary, the centre move on a line, and so on.

The first point, then, is to see the concept or the relation in question as the *invariant* of a dynamic pattern. The second is to organise the pattern in such a way that one invariant only can be discovered in it. The third is to enable our pupils to perceive it with their senses, supported by their minds, and to express what they have perceived in their own language. When this has been improved upon, the result may prove to be a statement significant for the geometrical education of our pupils. It then remains to give the proof, i.e., to relate the newly-acquired experience to earlier experiences in orderly fashion, although this will not necessarily increase the conviction of our pupils.

One or two examples will serve to illustrate how the method can be carried out in practice, firstly with the use of the blackboard and coloured chalk, and then with a film.

Let us suppose that our aim is to produce a dynamic pattern which will bring home the property of the angle in a semi-circle. Having drawn a semi-circle, diameter AB, we shall then draw any line L from the centre, meeting the circumference at M, but going beyond. We then choose two points, P, very near the centre O, and Q very far away, on the line L. Angles APB and AQB are obviously respectively almost a straight angle and a very acute angle. Points between P and O yield angles approaching 2 right angles, while points beyond Q yield angles approaching zero. It is a matter of a few minutes, in questioning the class, to obtain the certainty that the points on L inside the semi-circle yield obtuse angles, and those outside, acute angles. The point M of intersection is neither inside nor outside, and yields a right angle. The whole of the awareness of the value of the angle AMB is now linked with that of many others, by virtue of the dynamism of the pattern. The requirement of the syllabus has been met, but also this particular angle has been *placed* among an infinity of cases as the only one that is definite. For the teacher, two things remain to be done. One is to vary the semi-circle, and the other to vary the line L, so that the fact discovered is seen as an invariant of all these situations. Only then, when the result has been clearly stated by the pupils, can we know that it has become part of their experience and constitutes a geometrical fact for them. The formal proof is then the easy part of the work.

Our second example will be presented in the form of a film, of the type used by J. L. Nicolet and called *Animated Geometry*. The film is silent, only delineated figures are used, and there are no letters. The problem is to find the locus of points whose property is that the angles between the two tangents to each of two circles are equal. This is an interesting rider, and one not easily solved by pupils at the lower sixth form level. The film is animated in the

following way. First of all a point is chosen near one of the circles, one angle then being obtuse and the other acute, and the same is then done with the other circle. The angles are seen to be unequal, but by comparing the two cases we find that the relationship changes, and the idea begins to dawn that there may be an intermediary position in which the angles are equal. When one has been found, the question arises as to whether there are others, and if so, where. The locus is a circle containing the smaller of the given circles. The dynamic pattern then follows : the two circles remain centred where they were, but one increases and the other decreases, the locus gaining its full significance from the infinite number of cases of figures. The lessons contained in this film are obviously many, and we leave the reader to imagine their uses.

We have given these examples at length to illustrate the value and range of the idea of dynamic pattern. If more teachers of experience are prepared to put it into practice, it may well be that we shall find ourselves in possession of a truly exciting set of geometry lessons, lessons no longer divorced from our pupils' interests.

Nicolet's films are obviously much finer tools than the patterns we ourselves can draw, and they are also highly to be recommended for the aesthetic quality with which the genius of the Swiss pioneer has endowed them.

C. GATTEGNO.

2444. *A fable.*

Once upon a time there was a teacher who set his class an examination to perform. And when the youths had finished he marked their scripts. But at the end of his labours he found that, by evil chance, he had worked with a total of 99. And, being an industrious man, he converted all the marks into percentages.

So it was that a pupil with 58 marks gained

58·585858... per cent.,

and a pupil with 73 marks gained

73·737373... per cent.,

and others likewise.

And when the time was come that he should return the scripts to his class, being an honest man as well as an industrious, he confessed what he had done and delivered to them their marks in the form of percentages.

Until he came to one named Smith whose work was perfect, to whom perforce he had awarded the percentage

99·999999... per cent.

" So, Smith Minor," saith he, " Though I find no fault in you, yet your percentage falls short of the full total of 100. What say you? "

" Sir," saith Smith Minor, moved to anger, " I call that the limit! "

2445. *Interpretation of the correlation coefficient.*

The Lagrange identity

$$\Sigma x_r^2 \,.\, \Sigma y_r^2 - (\Sigma x_r y_r)^2 = \sum_{r<s} (x_r y_s - x_s y_r)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

gives an immediate intuitive interpretation of the correlation coefficient

$$r = (\Sigma x_r y_r)/\sqrt{(\Sigma x_r^2 \; \Sigma y_r^2)}$$

o

where (x_r, y_r) $(r = 1, 2, \dots, n)$ are coordinates of points in a scatter diagram referred to rectangular axes through their centroid. If all the points lie on a straight line, this line must pass through the centroid, and $y_r/x_r = y_s/x_s$ for every pair of points. Hence the right-hand side of (1) is zero, and $r^2 = 1$, so that $r = \pm 1$. In all other cases, $|r| < 1$. Further, since the right-hand side of (1) may be written

$$\sum_{r<s} x_r^2 x_s^2 \left(\frac{y_s}{x_s} - \frac{y_r}{x_r}\right)^2,$$

it will be very small if the points lie very nearly on a straight line through the centroid, for then $y_r/x_r \simeq y_s/x_s$ for every pair of points (x_r, y_r), (x_s, y_s). But if the right-hand side of (1) is very small, $(\Sigma x_r y_r)^2$ is nearly equal to $\Sigma x_r^2 \cdot \Sigma y_r^2$, and hence r^2 is nearly unity. Thus numerically small values of r occur only when there is appreciable scattering, and if r^2 is nearly unity there is close correlation.

D. A. T. WALLACE.

2446. *On the area of a square.*

Problem: To find the area of the region $|x| + |y| < 1$.

Solving for y gives $|y| < 1 - |x|$, i.e.,

$$-(1 - |x|) = y_2 < y < y_1 = 1 - |x|,$$

and the range of values for x is from -1 to $+1$.

Hence the area is $\int_{-1}^{1} (y_1 - y_2)\, dx$.

Omitting the limits, this gives

$$\int 2(1 - |x|)\, dx = 2x - 2\int x \operatorname{sgn} x\, dx = 2x - \int \operatorname{sgn} x\, d(x)^2$$

$$= 2x - x^2 \operatorname{sgn} x + \int x^2\, d \operatorname{sgn} x = 4 - 2 + 0 = 2,$$

on substituting the limits and using elementary properties of Stieltjes integrals. (The symbol sgn x denotes -1, 0, 1 for x negative, zero, positive, respectively).

It may be noticed incidentally that, for any real values of a and b,

$$\int_a^b |x|\, dx = \left[\tfrac{1}{2} x |x|\right]_a^b.$$

Two other ways of finding this formula are:

(*A*) By trial differentiation:

$$D(\tfrac{1}{2} x|x|) = \tfrac{1}{2} x \operatorname{sgn} x + \tfrac{1}{2}|x| = \tfrac{1}{2}|x| + \tfrac{1}{2}|x| = |x|,$$

and the fundamental theorem of integral calculus applies.

(*B*) By partial and Stieltjes integration:

$$I = \int_a^b |x|\, dx = x|x| - \int x\, d|x| = x|x| - \int x\, d(x \operatorname{sgn} x)$$

$$= x|x| - \int x \operatorname{sgn} x\, dx - \int x^2\, d \operatorname{sgn} x$$

$$= x|x| - \int |x|\, dx - 0 = x|x| - I,$$

giving $I = \left[\tfrac{1}{2} x|x|\right]_a^b$.

The geometrical interpretation as the area of the square whose equation is $|x|+|y|=1$ is left to the reader—who may also wish to try other similar methods of integration (e.g., using parametric representation or polar co-ordinates) for this problem and for the evaluation of the perimeter of the square. C. W.

2447. *Isosceles.*

Being impressed by the number of variants in the spelling of the word "isosceles", I began to collect them and arrange them in strict alphabetical order (so as to avoid duplicates). The number on my list is at present **129.** It is naturally increasing rather slowly now, but all possibilities are not yet exhausted and I have no doubt I shall find a few more yet.

Here are some other gleanings from scripts.

To the question "Reduce £28 per ton to pence per lb." the answers varied from 1,075,200 lb. for a penny to 15,052,800 pence per lb.

"A man cycles 440 yds. in 48 secs. Find his average speed in miles per hour." Answers given varied from 900 miles per hour (several times) to 1/691200 miles per hour, that is, 1 mile in approximately 79 years.

The average age of a class of 24 boys varied from 1 year 3 months to 46 years 2 months.

"They meet at 11.50, 39 miles apart."

"The rates were increased by £305 9s. 1d. in the £."

"If sold at £800, he would gain £1,300,000 more than if sold at £1000."

"If it did not stop, it would arrive at 3.45; it did, so it arrived at 3.7½."

"Angles in the same segment of a circle are equal because they are *suspended* by the same *cord*."

To solve the equation, $2\sin^2 x+\cos^2 x=1{\cdot}16$.

(i) $$\begin{aligned} 2\sin^2 x+\cos^2 x &= 1{\cdot}16 \\ \sin^2 x+\cos^2 x &= 1. \end{aligned}$$

Subtracting $$2x+x=\cdot 16,$$

and $$x=\cdot 053.$$

(ii) $$\sin^2 x+\cos^2 x=\frac{1{\cdot}16}{2}.$$

Thus (presumed by division!)

$$x=\frac{1{\cdot}16}{\sin^2 2+\cos^2 2}.$$

"In a circle, the chord CD is perpendicular to the diameter AB, intersecting it at N. Prove that the sum of the areas of the circles on AN, BN, CN, DN as diameters is equal to the area of the whole circle."

$$\begin{aligned} \text{Sum of areas} &= \tfrac{1}{4}\pi(AN^2+BN^2+CN^2+DN^2) \\ &= \tfrac{1}{4}\pi N^2(A+B+C+D). \end{aligned}$$

But A, B, C, D are all points on the circumference of the big circle. Hence the sum of the areas is equal to the area of the whole circle.

"O is a point inside a rectangle $ABCD$. Prove that

$$OA^2+OC^2=OB^2+OD^2."$$

If the equation is divided by O, its value is unchanged. Thus

$$A^2+C^2=B^2+D^2.$$

$A = B = C = D$ because the figure is a rectangle. Thus

$$A^2 = B^2 = C^2 = D^2$$

and so $$A^2 + C^2 = B^2 + D^2,$$

but this is equal to

$$OA^2 + OC^2 = OB^2 + OD^2.$$

G. W. CAUNT.

2448. *The cosine rule.*

In the discussion on "Trigonometry in the Main School" reported in the *Mathematical Gazette* for December 1951, it was pointed out that a common formulation of the proof of the addition formulae by means of the cosine rule was not valid for angles greater than 180°. Generalisation of the proof of the cosine rule for all angles, by one of the methods which follow, obviates this difficulty. Besides being useful at the beginning of a specialist course for revision of the cosine rule, it is a good example of the simplification caused by the use of the general angle and also in the second case of Pythagoras' Theorem applied to Trigonometry.

Figure 85 on page 65 of the Trigonometry report really suggests these methods. Both use the usual notation as to axes, coordinates, the definition of $\cos A = x/c$ and allow $\angle A$ to take any value.

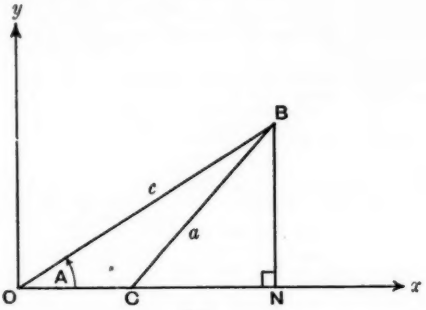

FIG.

Method 1. Taking B as the point (x, y),

from triangle BCN, $\quad a^2 = y^2 + (x - b)^2$;

from triangle OBN, $\quad c^2 = y^2 + x^2$.

By subtraction, $\quad a^2 = b^2 + c^2 - 2bx$

$$= b^2 + c^2 - 2bc \cos A.$$

Method 2. Taking B as the point $(c \cos A, c \sin A)$ and assuming that it is known that $\cos^2 A + \sin^2 A = 1$, from triangle BCN,

$$\begin{aligned} a^2 &= (c \sin A)^2 + (c \cos A - b)^2 \\ &= b^2 + c^2(\sin^2 A + \cos^2 A) - 2bc \cos A \\ &= b^2 + c^2 - 2bc \cos A. \end{aligned}$$

A. HURRELL.

2449. *The flexagon and the hexahexaflexagram.*

The Flexagon and the Hexahexaflexagram (called hereafter F and H) are strips of paper, in the form of a row of equilateral triangles, which are joined end to end and can then be folded into the shape of a regular hexagon in several different ways. In each method of folding a different set of faces of the triangles is uppermost, so that, if these sets are numbered or coloured differently, a different set of numbers or colours is visible at each folding.

F was discovered in 1940 by a group of scientists (R. Feynman, A. H. Stone, B. Tuckerman and J. W. Tulsey). H was the subject of an article by Miss Margaret Joseph in *The Mathematics Teacher* for April 1951. So far as is known, no account of either F or H had appeared previously, and this is the first description of F to be printed.

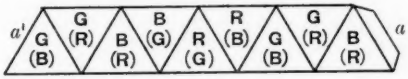

FIG. 1.

F is the simpler of the two. It consists of 9 equilateral triangles in a row (Fig. 1). A flap is added to one of the end triangles for glueing. The triangles are coloured green, blue or red both on the front and the back : in Fig. 1, the letters denote the colours on the front, while the letters in brackets denote the colours on the back. The flap (a) is glued to the side (a') after the strip has been given one and a half twists, that is, one end rotated through 6 right angles relative to the other end. The surface formed is a one-sided surface like a Möbius band, but with an extra twist.

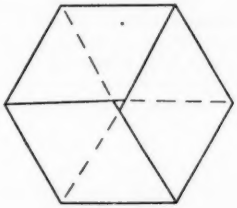

FIG. 2.

Now that F is joined up, it will be found that we can press it flat (folding only along the lines where the triangles join) to form a hexagon so that, for instance, 6 green faces are on top and 6 blue underneath, while all the red faces are hidden. A very easy and rapid adjustment, something like turning a bag inside out, will change the folding to either of the other two possible ways. Fig. 2 is a sketch of the completed flexagon.

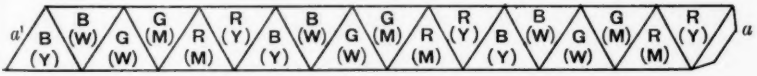

FIG. 3.

H is a more ambitious affair. We require 18 equilateral triangles coloured in six colours : blue, green, red, yellow, white, mauve (Fig. 3). The strip is given three complete twists (12 right angles) before being glued up. Thus it is a two-sided surface, a cylindrical strip with three twists. When pressed flat so that the blue faces, for instance, are uppermost, the effect is like F except that every alternate blue face is part of a hinged triangular flap consisting of 3 triangular faces on top of one another (Fig. 4). If each of these three flaps

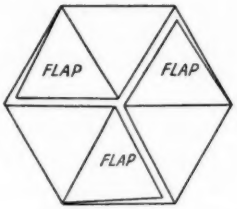

FIG. 4.

is turned over, then the 6 upper faces will be all mauve. If H is turned completely over the faces may be green or yellow according to which way the flaps are turned. The remaining two colours, red and white, are hidden in this folding : to exhibit them, H must be turned inside out in the same way as F was, when either blue and mauve or green and yellow will be hidden.

The colouring and method of folding H as here given are slightly different from those given by Miss Joseph. This revised method is due to A. R. Pargeter and seems to be easier to follow than Miss Joseph's method. It also has the advantage of emphasising the connection between F and H.

It has been suggested that H is a particular case of the rotating rings of tetrahedra mentioned in Rouse Ball and Coxeter, *Mathematical recreations and essays*. Certainly, if we make a ring of 9 tetrahedra and then remove two faces of each tetrahedron, we obtain a surface formed by a row of 18 equilateral triangles joined in a ring, but this has only two twists and is not H. It could be turned into H by cutting, inserting another twist, and rejoining.

F. G. MAUNSELL.

2450. *Random digits.*

Further to Note 2267 (XXXVI, No. 315, February 1952), I have taken a sample of 700 digits at random. I obtained the numbers from Chambers's seven-figure logarithms and chose the seventh figure of the logarithms of all numbers ending in 4 between 20004 and 26994, and obtained the following frequency distribution :

Digit	0	1	2	3	4	5	6	7	8	9
Frequency	67	74	74	75	70	52	70	82	63	73

B. M. MCKENZIE.

2451. On Note 2410.

In Note 2410 Mr. Goldby described a simple method for approximate estimation of standard deviations. In 1949 I published in *Nature*, Vol. 164, p. 718 an identical method. It is so simple that it must have occurred to many people but I had not found any previous reference.

G. S. SMITH.

REVIEWS.

Algebraic Geometry. By S. LEFSCHETZ. Pp. ix, 233. 30s. 1953. Princeton Mathematical Series, 18 (Princeton University Press; Geoffrey Cumberlege, London)

Any book by Professor Lefschetz is bound to be of interest, and to those who know the author it will come as no surprise that in his seventieth year he can produce a book which is as vigorous, as stimulating, and, if I may say so, as infuriating as any he has ever written.

This book can only be described as typical Lefschetz. It moves at break-neck speed, throws out ideas at an incredible pace, and passes on to pastures new before the reader has time to look about him, and long before the end the reader finds himself breathless, but fully resolved not to give up the chase. It is, I suppose, theoretically possible to stop at any particular point and ponder on what one has read, but Lefschetz's powerful personality comes through the printed page so strongly that in practice it is no more possible to stop in the middle of the book than it is to dam the tide of ideas which one encounters in conversation with the illustrious author.

The manner in which the book has been written is of a piece with Lefschetz's attitude to geometry. Although by the late twenties or early thirties his research work had become almost entirely concerned with pure topology, Lefschetz has never abandoned his interest in algebraic geometry and he has kept turning his mind back to the knotty problems of this subject. He has published very little on algebraic geometry since 1930, but more than one geometrical idea of prime importance has come since then from ideas given by him to pupils and colleagues. When, in the later thirties, algebraic geometry appeared to be in the doldrums, he courageously strove to revive interest in it by giving courses of lectures at Princeton; and now that the subject has regained its former proud position in mathematics no man is more pleased than Lefschetz. The present book owes its origin to the lectures given between 1935 and 1938.

These lectures (only available at the time in mimeographed form and for long quite unobtainable) dealt with the theory of curves over any ground field of characteristic zero. Lefschetz's method of treating curves was to use formal power series in place of convergent power series in a complex variable, and the message of his lectures was that this method was capable of extending all the algebraic geometry over the complex field to the theory of curves over any field of characteristic zero. I am not quite sure when he first propounded this doctrine, but for many years he has held the view that, in a sense, geometry over any field of characteristic zero can be reduced to geometry over the complex field. This point is argued in an appendix to the present volume, and is a key to the whole book.

The lectures form the substance of Chapters V-VIII. In Chapters I-III a rapid account is given of the basic ideas of the geometry of varieties in a space of n dimensions over an arbitrary ground field. Naturally much of this covers ground which is now familiar, but there are many places where ideas are given new twists, and the survey leaves much room for thought. Chapter IV is devoted to the theory of formal power series, and provides a useful account of this topic of which easily available accounts are at present few. Then in Chapter IX, the last chapter, Lefschetz returns to another old love, the transcendental theory of surfaces, with particular reference to the theory of the base. This theory has been the subject of many books and papers, and to compress it into one chapter Lefschetz has had to quote extensively from the works of Picard and others, and, indeed, it is only understandable by those

who are already familiar with the background. But it represents his latest thoughts (I refuse to say his last thoughts) on a subject of which he is one of the great masters, and it will no doubt be a subject of careful study by all who are interested in the field. The main point of the chapter is that much of Picard's work, particularly in connection with the invariant ρ_0 of a surface, can be expressed in purely algebraic form, and it is this central theme that gives the last chapter of the book a unity with what precedes.

And so we have a book entirely typical of the grand old warrior, and one that many, particularly those who know and reverence the author for his many endearing qualities, will greatly treasure. W. V. D. HODGE.

Theory and Applications of Distance Geometry. By L. M. BLUMENTHAL. Pp. xi, 347. 50s. 1953. (Oxford)

This book is an account of work on the geometry of metric spaces inaugurated by Menger and largely developed by the author. A metric space is a set of points such that, for every pair of points p, q a distance function (pq) is defined such that (i) $(pp)=0$, (ii) $(pq)=(qp)>0$ if $p\neq q$, (iii) $(pq)+(qr)\geqslant(pr)$. Distance geometry is essentially the study of properties of a space invariant under distance preserving homeomorphisms.

The first few chapters of the book are devoted to the general theory, which is carried to the point at which the notions of arc-lengths and curvature can be defined (in a situation far more general than that treated in differential geometry). The main body of the book centres round a discussion of two questions. First, how may a given space (e.g. a euclidean, spherical, elliptic or hyperbolic space of n dimensions) be characterised in terms of metric geometry—that is, in terms of relations between the distances of sets of points in it? Secondly, given a point set in a semimetric space, under what circumstances does it admit a congruent embedding in a given metric space? These problems appear, at present, to be intractable in their most general form, and the book gives a wealth of results dealing with restricted forms of the problem —for instance, when the space into which the point set is to be embedded is euclidean or spherical. An important role is played by determinants of a particular type, and many conditions are expressed in terms of these determinants. The material here is so tightly knit together that it is hardly possible to quote actual results.

The last section of the book deals with various applications to determinant theory, lattice theory and linear inequalities. To the reviewer it seems that this section might be more accurately described as a collection of curious results, of a very special character, which appear as by-products of the theory previously developed. This is perhaps a question of taste, and it is possibly of interest to quote one of the determinant theorems, of which apparently no direct algebraic proof has been obtained. The theorem is striking, but obviously very special in its nature.

Let $\Delta=|r_{ij}|$ be a symmetric m-rowed determinant, where $m>n+3$, n being a positive integer. Suppose that $r_{ii}=1$, $-1<r_{ij}<1$, $(i\neq j,\ i,\ j=1,\ 2,\ \ldots,\ m)$. Suppose further that (i) every principal minor of Δ of order less than $n+2$ is non-negative, (ii) every principal $(n+2)$-rowed minor of Δ is zero, (iii) some principal $(n+3)$-rowed minor is not zero. Then, if $i\neq j$, $|r_{ij}|=1/(n+1)$, and by multiplying certain rows and the corresponding columns of Δ by -1 we can ensure that each non-diagonal element of Δ is $-1/(n+1)$. Further, every k-rowed principal minor of Δ has the value

$$(n+2)^{k-1}(n-k+2)/(n+1)^k, \text{ for } 1\leqslant k\leqslant m.$$

The presentation of the theory is formal and very concise; the exposition is clear, but attentive study is required to follow the details of the proofs,

which are sometimes involved. The book should prove a useful guide to those interested in this rather specialised field. The printing is excellent.

J. A. T.

Cours de Géométrie Infinitésimale. Fascicule I, Vecteurs et Tenseurs, Théorie Élémentaire. By GASTON JULIA. Second Edition, pp. xv, 103. 2000 fr. 1953. (Gauthier-Villars, Paris).

The first edition of Julia's " Cours de Géométrie de l'École Polytechnique " was published about 15 years ago. In the second edition the author has made many changes based on experience gained since that time, and the result is effectively a new work.

In accordance with the spirit of the first edition, the course has two objectives—first to familiarise the student with the methods of analytical geometry, including vectors and tensors, to train him to master these methods, and to demonstrate how these can be applied to other branches of mathematics. The second objective is to develop the student's intuition, his judgment and his appreciation of mathematical elegance, by comparing the advantages of different methods and in selecting the most appropriate method for solving a specific problem.

The book reviewed forms chapter I of the second edition, and deals with the elementary theory of vectors and tensors in Cartesian coordinates. This chapter sets a very high standard for the rest of the work, as it contains one of the most lucid and interesting expositions on the subject.

There are a number of trivial misprints—e.g., p. 13 line 4, for V read $\overline{V}$; p. 16 line 6, for $\overline{O}$ read O; p. 18 line 2, for $a' = b'$ read $a = b'$; p. 19 the point B is omitted from fig. 10; p. 32 line 1, for $\overline{V}u_0$ read $\overline{V}(u_0)$. But generally the printing is good and the lay-out is excellent.

It is very unfortunate that such a beautifully written book should be so expensive. I doubt whether many students in this country will pay around 47s. 6d. for this first chapter of about 100 pages between paper covers.

T. J. WILLMORE.

Ludwig Schläfli (1814-1895). Gesammelte Mathematische Abhandlung. II. Pp. 381. Sw. fr. 56.15. 1953. (Birkhäuser, Basel)

The first volume of Schläfli's collected works was reviewed in No. 309 of the *Gazette*, p. 239. Of the seventeen papers in the second volume, the one which is perhaps of most interest today is his memoir on the twenty-seven lines on the general cubic surface, in which the " double-six " appears for the first time, and the notation a_{ij}, b_{ij}, c_{ij} still current for the lines is introduced. This paper also obtains the division of real cubic surfaces into species according to the configurations formed by their real lines. Another paper describes the various types of singular cubic surface. These two papers, which were published in an English translation by Cayley, earned for Schläfli the Steiner prize of the Berlin Academy in 1870, and have obviously influenced the account of the subject given by Salmon in his treatise. There are also included the short accounts of the results contained in his " Theorie der vielfachen Kontinuität " (already reprinted in the first volume) which were published during his lifetime, and a long paper dealing with the resultant of a system of algebraic equations (dating from 1851), which well exhibits the author's technical virtuosity, albeit his conception of a resultant, as van der Waerden points out in an editorial note, is imprecise by present-day standards.

The production maintains the high standard set by the preceding volume; the paper is of a quality rarely seen in our own books now-a-days.

J. A. T.

Theorie der geometrischen Konstruktionen. By L. BIEBERBACH. Pp. 162. Sw. fr. 15.50; geb. sw. fr. 18.70. 1952. (Birkhäuser, Basel)

This book gives a comprehensive and authoritative account of the theory of geometrical constructions. It is thoroughly modern in outlook, clear and lively in exposition, and it should appeal both to the learner and the expert. The author develops the subject with great skill, adopting an arrangement of the material by which he is able continually to lead the reader to the most recent work, and frequent statements of questions as yet unsolved or unconsidered add to the interest and usefulness of a book in which so much of the subject matter is now classical.

The treatment is elementary, and little previous knowledge is explicitly assumed; but difficulties are never avoided, and while most of the geometry and trigonometry are within the understanding of a good VIth form boy, parts of the algebra are more difficult, and would be appreciated only by the more experienced reader.

The exposition is based on the discussion of the constructions which are possible with various instruments used in precisely stated ways. The first 12 of the 30 sections are concerned with constructions of the first and second degrees, ruler, ruler with fixed circle and its centre, ruler and compass, compass alone, parallel and angle rules, and so on, with recent extensions by Hjelmslev and others. There is an interesting theorem due to F. Bachmann (1949) on the points constructible with a certain use of the set square, and an algebraic discussion of the Hilbert and Bachmann constructions.

The next part, to section 26, deals chiefly with third and fourth degree constructions. An introduction to the relevant algebra is followed by a systematic account of methods of solution of the two problems on which ultimately all such constructions depend, namely the trisection of an angle and the finding of the cube root of a number. These methods have their source in the classical use of the conchoids of the straight line and circle, the cissoid, and the intersection of a circle with a given conic, but the presentation is original and stimulating, and the procedures described involve only simple extensions in the use of the ordinary instruments. The book concludes with a proof due to Gelfond and Siegel of the transcendence of e, and some account of constructions on the surface of a sphere. A useful appendix contains notes on the text and bibliographical references, mostly to recent work, and there is an index.

The book is pleasingly set out in clear type, and the diagrams are good; a few minor misprints were noticed ((16) on line 21, p. 104, should be (8)).

The author and his publisher may be congratulated on this valuable contribution to the literature on geometrical constructions. E. D. C.

Introduction to Dynamics. By L. A. PARS. Pp. xxii, 501. 31s. 6d. 1953. (Cambridge Univ. Press)

This volume written by Mr. Pars, a Fellow of Jesus College Cambridge and University lecturer in mathematics, sets out *ab initio* to develop the study of two dimensional dynamics of the particle and rigid body up to (but not including) Lagrange's equations. Whilst in substance the book is elementary, the treatment, with its accent on the *minutiae* of rigour together with the inclusion of certain more advanced topics such as inverse cube law in central force problems, make it an ideal textbook for the first year university student and the advanced student at school. The reader is warned in the preface that a fair knowledge of the calculus is expected, and indeed, is essential for the proper understanding of the subject. Here the reviewer is heartily in accord with the author.

The work commences with three introductory chapters on vectors and the fundamentals of Newtonian mechanics, followed by four chapters on one dimensional problems concerned with the single particle. The author points out that this apparent emphasis is justified by the fact that many of the ideas explained in these initial chapters occur later on. A great deal of attention is paid to the consideration of the simple energy equation in the form $\frac{1}{2}\dot{x}^2 = \phi(x)$ It is shown how to predict the general nature of the motion from the graph of $\phi(x)$ and how to obtain the equilibrium positions and to deduce the stability of these positions. Although there is nothing new in the methods employed, the account given in this volume is the most complete and satisfactory that the reviewer has seen in an elementary textbook. The eighth chapter concerns the rectilinear motion of a system of two particles and concludes with an account of impulses and collisions, whilst the ninth chapter concludes the rectilinear dynamics by a short discourse on variable mass. The next three chapters of the work deal in order, with the motion of the particle in a plane, conservative fields, two dimensional motion with constraint, and the theory of projectiles. In chapters 13 to 15 there is a thorough account of central orbits including the disturbed circular orbit and the inverse cube law, etc., whilst chapter 16 concerns itself with motion on rough curves and two dimensional motion in resisting media. Next we have a discussion on the motion of two particles in the two dimensional plane followed by a full account of the kinematics of a lamina in chapter 18. The latter ends with the Coriolis theorem and its application to the discussion of the motion of a bead on a rotating wire. In chapter 19 there is a brief summary of the main results in statics necessary for the solution of problems concerned with the motion of the rigid body in general, followed by an account of D'Alembert's principle, whilst chapter 20 gives an account of motion about a fixed axis. Chapters 21 and 22 are then devoted to the continuous and impulsive motion respectively, of the rigid body in a plane and is followed by the dynamics of a system of particles in chapter 23 and a short chapter on the principles of momenta. The book ends with an account of the impulsive motion of a system and finally, on dimensions.

The volume is original both in its order and in the mode of presentation of its topics. It is well planned and well written, a book which can be recommended without any reservations. The basis of the book is a sound logical development from fundamental principles clearly stated and defined, so that the reader has the best possible introduction to the further developments in the subject and its allied branches.

There are numerous exercises in the text to test the reader's understanding and an adequate index. The printing is excellent and the book is attractive in appearance.

J. WILLIAMS.

Scientific Explanation. A study of the Function of Theory, Probability and Law in Science. By R. B. BRAITHWAITE. Pp. xii, 376. 40s. 1953. (Cambridge University Press)

In 1946 the author, who is now Professor of Moral Philosophy in the University of Cambridge, gave the ninth Tarner lectures and delivered the Presidental Address to the Aristotelian Society. This book is based on the former and incorporates, in Chapter X, the latter.

The contents of the volume might be divided into three parts, which, though connected, have different foci. The first deals with models, theoretical terms, and deduction (four chapters), the second introduces probability and statistical hypotheses (three chapters) and the third analyses induction, causality, and teleological explanation (four chapters). The stress is everywhere not on how science *should* work, but on how it *does* work.

To explain the function of a model within a scientific theory, the author exhibits a calculus, relating each principle of deduction to a corresponding rule of symbolic manipulation. "A calculus, considered in isolation from any interpretation of it, is a game for one player, played with marks on paper" (p. 25). He illustrates how the same calculus can be the object of different interpretations and illustrates the status of theoretical concepts. The difference between the use of calculi in physics and in mathematics is that in the latter discipline all conclusions follow logically from the assumptions, which are themselves largely arbitrary, whereas in physics it is actually the set of lowest order conclusions which is given first—describing observations—and the higher level statements are derived, and expressed in theoretical terms.

Bertrand Russell's doctrine of logical constructions is criticised, following F. P. Ramsey, and the author's objection is illustrated by ingeniously constructed, simple "scientific theories". The conclusion is, that "while the theoretical terms of a scientific theory are *implicitly defined* by their occurrence in initial formulae in a calculus in which there are derived formulae interpreted as empirical generalisations, the theoretical terms cannot be *explicitly defined* by means of the interpretations of the terms in these derived formulae without the theory thereby becoming incapable of growth" (p. 77).

With Chapter V we enter the domain of probability and statistics. He would indeed be an unrealistic logician who ignored the change of emphasis in natural philosophy due to the emergence of quantum theory and its derivatives. The argument follows the Neyman-Pearson theory of testing statistical hypotheses and develops into an account of Wald's theory of decision functions. This is a very readable presentation of the fundamentals, and the spirit of Wald's theory is correctly reflected by connecting it with von Neumann's Theory of Games. The minimax solution, which produces the best obtainable result if the opponent (in this case "nature") does his worst, is oddly called the "Prudential policy". The author's line of approach is admirably stated at the end of Chapter VII: "it would seem less in accordance both with the principle of empiricism and of that of parsimony to justify the choice between scientific hypotheses by attributing non-empirical prior probabilities to these hypotheses than to make use of those evaluative criteria for causes of action which we employ every day in our ordinary life" (p. 254).

Incidentally, the author gives a simple and useful geometrical method for finding optimal strategies in a game in which one of the opponents has only mixtures of two pure strategies at his disposal, thereby extending the method of von Neumann himself. Since in the book this is only a side-line, the full implications of the method are only hinted at in a footnote on p. 239, but this is unfortunately marred by a slip in the description. The question of an apparently infinite regress of statistical tests (every acceptance or rejection of a probability statement being by its nature subject to an appeal to higher-leve probability statements) is tackled and resolved.

In his preface, the author states that his account of how probability is used in a science, and of the relation between theories and models, is original. He also admits that he is not satisfied with what he says about induction in Chapter VIII, although it is still worth saying. This is the chapter which the reviewer found it most difficult to appreciate, mainly because about one half of it is taken up by the refutation of an alleged circularity in the predictionist justification of induction, which approves of it if and only if it is based on a policy which has been effective in the past. The account of teleological explanations, i.e. those which contain references to a future state of affairs, is again claimed to be original. In the author's view such references do not present any difficulty when they concern "intentional human action in terms of a goal to the attainment of which the action is a means" (p. 324). Such

explanations can be seen to be disguised causal explanations. Those which cannot be thus replaced are justified by the fact that they perform " both of the functions characteristic of scientific explanations—of enabling us to appreciate connections and to predict the future " (p. 335). This expresses best the author's attitude to what is required of science.

The book, with its handy size, on good paper and in type and print confirming the reputation of the publishers, is an excellent acquisition for the bookshelf of anybody who wishes to be aware of those fundamental problems of science about which the author felt that he had something to say. In his own words (p. xi), used about the examples which he has constructed, if the book is not " a contribution to ' learning and research ' (it is) a definite contribution to ' education ',' and it offers even more than this to the specialist. S. V.

Pages choisies d'analyse générale. Par M. FRÉCHET. Pp. 213. 2000 fr. Collection de logique mathématique, A III. 1953. (Gauthier-Villars, Paris)

As the title suggests, this is a collection of (extracts from) papers published by Professor Fréchet during the past fifty years, most of them in fact belonging to the years 1904–1924. Though some of the extracts are now chiefly of historical interest the collection is impressive testimony to the power of the ideas contained in Fréchet's doctoral dissertation of 1907, and some of the early papers, working out the consequences of these ideas, are remarkably " present day " in spirit and outlook.

The papers fall under three main subdivisions : function spaces, functional analysis and generalised analysis. A distanced function space in Fréchet's sense is a collection of elements with each pair, a, b of which is associated a number (a, b), called the *distance* between the elements, satisfying the following conditions:

(i) $(a, b) = (b, a) \geqslant 0$;
(ii) $(a, b) = 0$, if and only if a and b denote the same element ;
(iii) $(a, b) \leqslant (a, c) + (c, b)$;
(iv) the necessary and sufficient condition for a sequence of elements $a_1, a_2, a_3, \ldots$ to converge towards an element a is that (a_n, a) converges to zero. (A space satisfying condition (iv) is said to be *compact*).

The elements may be of any kind, though the cases considered are those in which the elements are function or sets. Amongst many important examples given of distanced compact function spaces are (a) the class of continuous functions of one real variable, (b) the class of functions of the class L^2 (that is, the functions $f(x)$ such that $f(x)$ and $f(x)^2$ are both integrable in the Lebesgue sense), (c) the class of complex functions holomorphic in the interior of a finite region R bounded by a simple or multiple curve. In example (c), for instance, the distance between two elements $f(z)$, $g(z)$ is defined as follows. We introduce an expanding sequence of regions R_n, each containing its successor and contained in R, and such that every region R' completely inside R lies inside R_n for some n ; let u_n be the maximum modulus of $f(z) - g(z)$ on the boundary of R_n. Then the limit of the convergent series

$$\frac{u_1}{1+u_1} + \frac{1}{2!}\frac{u_2}{1+u_2} + \frac{1}{3!}\frac{u_3}{1+u_3} + \ldots$$

is taken to be the distance (f, g), which is readily shown to satisfy the conditions (i)–(iv) above, if we take the convergence of $f_n(z)$ to $f(z)$ to be uniform in any region R' completely contained inside R.

The object of defining distance in a function space is of course that thereby many familiar properties of the classical theory of functions may be extended

from the linear point continuum to a space of general elements. Consider for instance the space C of functions $f(x)$ of one variable continuous in a closed interval (a, b), and let U_f be a functional defined over this space, that is, a number corresponding to each function f of the space (such as for example the integral of $f(x)$ from a to b). The functional U_f is said to be continuous if U_ϕ tends to U_f if $\phi(x)$ tends to $f(x)$ uniformly in (a, b). Then it follows from the fact that C is a distanced compact space that if U_f is continuous in C then it is uniformly continuous in C, and bounded and attains its bound in C.

The class C may of course be considered as the class of all infinite sequences of numbers, x_n, such that $\Sigma {x_n}^2$ converges, since any continuous function of one variable determines and is determined by the sequence of its Fourier coefficients (by the Riesz-Fischer theorem).

Fréchet shows that the properties of differentiability and integrability are definable for functionals, and establishes Taylor's theorem with remainder in the form

$$U_{f_0+\phi} = U_{f_0} + \partial_\phi U_{f_0} + \frac{1}{2!}\partial^2_{\phi^2} U_{f_0} + \ldots + \frac{1}{n!}\partial^n_{\phi^n} U_{f_0} + \theta_\phi$$

where $\partial_\phi U_{f_0}$ is the differential of the functional U_f, that is, a functional linear in ϕ which differs from the difference $U_{f_0+\phi} - U_{f_0}$ by a quantity infinitely small with respect to the maximum of $|\phi|$ in the interval under consideration.

Changing the notation, we denote by $F(A)$ a functional defined for each element A of an abstract set E, belonging to an additive family of sets $\mathcal{F}$, and by $f(e)$ an additive function of sets defined on $\mathcal{F}$. Fréchet shows that $f(e)$ may be expressed as the difference of two positive functions $\phi(e)$, $\psi(e)$ also additive on $\mathcal{F}$. We divide E into a finite sum of disjoint sets $E_1, E_2, \ldots$ and form the sums

$$S' = \Sigma M_n \phi(E_n),\ s' = \Sigma m_n \psi(E_n),$$

where M_n, m_n are the bounds (supposed existent) of $F(A)$ in the set E_n. The sum S' is bounded below and the greatest lower bound of S' for all subdivisions of E is called the upper integral of F on E with respect to ϕ and $\mathcal{F}$. Similarly the least upper bound of s' is the lower integral of F on E with respect to ψ. If the difference between the upper integral with respect to ϕ and the lower with respect to ψ is equal to the difference between the lower integral with respect to ϕ and the upper integral with respect to ψ, their common value is called the integral over E of $F(A)$ with respect to $f(e)$ and relative to $\mathcal{F}$. Finally the definition is extended to cover the case when the function $F(A)$ is unbounded. R. L. GOODSTEIN.

Constructive Formalism. Essays on the foundations of mathematics. By R. L. GOODSTEIN. Pp. 91. 15s. 1951. (University College, Leicester)

In this very interesting and original book R. L. Goodstein presents the leading philosophical ideas underlying his foundational investigations, as also the main results obtained by them.

His attitude concerning method in mathematics is that of a strict finitism, conformable to Hilbert's "finitem Standpunkt"; but whereas Hilbert restricts the requirement of finitism to metamathematics, Goodstein wants to have a formal system with a finitist interpretation. So he has been led to treat analysis in the frame of recursive number-theory, which, by the elimination of the logical symbols, was reduced—independently by the author and by H. B. Curry—to a pure equation calculus, i.e., a calculus all of whose formulae have the form $a = b$. Goodstein succeeds in showing that a great portion of analysis can be developed in this way. As is to be expected, many features here resemble Brouwer's analysis. However Goodstein does not use the concept of

Wahlfolge, and whereas Brouwer characterises a continuous function by co-ordination of rational intervals, Goodstein defines it as a sequence of rational functions (of rational arguments).

A method is also explained for treating the theory of transfinite numbers (to a certain extent) in the constructive finitist frame. The point here is that Cantor's theorem that any decreasing sequence of ordinals terminates, can be translated (for the ordinals here considered) into a constructive number-theoretic statement.

A closer discussion is devoted to Gödel's result about formal undecidability and to the problem of proving consistency. The reader could get the impression that the consistency problem arises only for classical non-finitist mathematics. However it must be noted that the question of consistency also exists for the finitist formalism. Thus in particular, if recursive definition is taken as a conventional method of introducing functions, then *a priori* the possibility is not excluded of deriving by the rules of the finitist calculus equations such as $0 = 1$. It is true that we can, by finitist reasoning, show that this cannot occur. However such a reasoning then will presuppose the possibility of calculations with number signs of arbitrary length, and so we become aware that already in the usual " finiten Standpunkt " we have a kind of theoretical idealisation of what can really be effected by concrete operations.

The material mathematical presentations are embedded in the philosophical discussions on method, in which the author fights for his constructivist point of view. In discussing the controversy between " formalists " and " constructivists " Goodstein tends to make formalism appear as directly contradictory. " The formalists " he says " affirm the possibility of completing an endless process " ; but " a completed infinite process is a contradiction in terms " ; for " an infinite process is by definition a process in which each stage . . . is followed by another stage " (p. 17, 13). However the said definition only excludes the assumption of an infinite process with a final stage, but not the consideration of an infinite process as a whole structure. The word " completed " in this respect is ambiguous. By the way, the use of the expression " formalist " for the opponent of the constructivist or the finitist (which has originated from the intuitionistic school), is somewhat seductive. In fact the view that we can regard an infinity as a whole does not include a formalist view on mathematics.

It seems that there is no possibility of proving finitism to be logically necessary. But neither can finitism be refuted, and the reasonings of the author, made to secure this, have their full strength ; he discusses the Zeno paradox from which it could be thought to follow that the existence of motion proves the existence of completed infinities. Goodstein's way of resolving the paradox (which is in full agreement with the view uttered in the *Grundlagen der Mathematik*) consists in distinguishing a literal and a metaphorical use of the expression " moving from one point to another ".

However, finitism is only one directive line of Goodstein's philosophy of mathematics ; its peculiar character mainly consists in the formalist views. In a parallel to Hume's duality of matters of fact and relations of ideas, he distinguishes among the sentence forms of a language the " matter-of-fact " and the " sign-convention " sentence forms. The sign-convention sentences are said to " contain the whole of mathematics " ; such a sentence is " neither true nor false, but simply is, or is not, one of the rules or a deduction from the rules of a certain arithmetic " (p. 21). The equation calculus " presents mathematics as a system of transforms in which neither axioms nor symbolic logic play any part " (p. 32). Indeed, it is an important result that mathematics, or at all events, a great deal of mathematics, can be presented in this way.

But in this form of presentation, what is properly mathematical appears only implicitly, namely by the facts on derivability which hold independently of conventions. It might here be noticed that the proper, interesting arithmetic does not consist in mere handling according to the rules of the equation calculus, but in proving general theorems on it, which has to be done outside of that calculus itself.

If we do not tend to reduce mathematics to logic, as the author surely does not, then it seems that we have to recognise mathematical structures and relations as a kind of facts *sui generis*. It is that kind of facticity by which the author himself is enabled, in developing his theory of the equation calculus, to give intuitive proofs.

On the conventionalist view of mathematics which the author defends, the problem of application of mathematics to reality becomes particularly urgent. To this problem, and more generally to that of the relation of language to reality, the last chapter, Language and Experience, is devoted. Here the author does not intend to treat the question systematically, but rather contents himself with illustrating his views by discussing, in a somewhat aphoristic way, some principal questions and some doubts. The philosophical attitude here expressed is radically critical of, if not even opposed to, theoretical scientific speculation in general. Important concepts and successful hypotheses are declared to be " only a new language " or mere " linguistic conventions ". After all, the reader might ask himself, are not many of the author's theses also " only " a certain philosophical language, of which the suitability might be questioned. Thus, in particular, what do we gain by speaking of mathematical propositions as sign-convention sentences? Do we understand better by this that we have theorems like the fundamental theorem of algebra or geometrical facts like that of the existence of six regular polytopes in four-dimensional euclidean space? Is it suitable to strengthen the necessary distinction between mathematical facts and those of natural science to a full rupture? In fact, to a mathematician it is not indifferent that the theorems of number theory can be tested by concrete computations and that geometrical curves and surfaces can be found, in an elegant and impressive way of approximation, realised in concrete objects. Is not what Galileo expressed by saying that mathematics is the language of nature important for the significance of mathematics? Would it not be more suitable to have a philosophic language which, like that of F. Gonseth's philosophy, states different aspects of mathematics, inter-related one with another?

Certainly the philosophical views of the author have directed him to carrying through, in so consequent a way, his finitist formalism. However, what he has so obtained is independent of that philosophy, and can also be appreciated by someone who does not adhere to the conventionalist views and does not believe that the equation calculus is all of mathematics.

On the other hand, the philosophical discussions given in different parts of the book, and especially in the last chapter—they are surely intended for people already sufficiently experienced in science—will give to such a reader a good stimulus to thought over many points, and generally will be wholesome for all those who tend to be too dogmatic in science. PAUL BERNAYS.

Logic for Mathematicians. By J. B. ROSSER. Pp. xiv, 530. 85s. 1953. (McGraw-Hill)

Of the many books on logic the most suitable for mathematicians, Professor Rosser thinks, is still the great three-volumed work of Russell and Whitehead, and his aim in the present text is to use new techniques, and improved symbolic apparatus, invented since the writing of *Principia Mathematica*, to

condense most of it into some 500 pages. This aim is not of course a new one, since Quine's *Mathematical logic* had much the same objective, and some comparison of the subject of this review with Quine's much slimmer volume is inevitable.

On the question of size, *Logic for mathematicians* is larger, not from any lack of conciseness, but on account of the detailed development of the theory of transfinite cardinals and ordinals into which Quine's book does not enter. Topic for topic, the extent of each account is comparable. In treatment Rosser's book will certainly have greater appeal to a reader whose training has been in mathematical analysis, whereas Quine's will appeal more to someone with a philosopher's background. This is perhaps the only sense in which Rosser's title is justifiable, for it is difficult to see that mathematicians have any special need of formal logic. Certainly it is very helpful for a beginner in the theory of series and function theory to be able to formulate definitions of convergence and divergence, continuity and discontinuity, uniformity and non-uniformity in symbolic form, but he will not need to read a text on mathematical logic to learn how to do that. There is nothing in *Logic for mathematicians*, as there is for instance in Robinson's *Metamathematics of algebra*, to show that symbolic logic is a source of discovery for other branches of mathematics. What in fact we find is an extremely careful presentation of a systematic development of abstract set theory as a branch of logic.

The opening chapters on the statement calculus and the restricted predicate calculus are particularly attractive and constitute one of the best introductions to the subject that I know. The formulation has many points of novelty, for instance in the wise use of the deduction theorem (to keep as close as possible to " natural " mathematical reasoning) and the early introduction of what may be called " specimen values ", that is, letters denoting values of variables for which certain relations hold.

A disturbing feature of these opening chapters, however, is the number of different roles which the same letters are required to play. There are bound variables (operated upon by universal and existential operators), so-called free variables (unknowns and inderminates) which play both the part of " any number " and " some unknown number ", as well as the " specimen value " variables already referred to, all denoted by the same letters. While there is no confusion in using the same letter x in both the contexts

$$(x^2-1)=(x-1)(x+1), \quad (x)\,((x^2-1)=(x-1)(x-1))$$

since each statement is a consequence of the other, it is certainly not safe to use it again to stand for one of the roots (unspecified) of the equation

$$x^2-4x+3=0.$$

Later in the book a descriptions operator is introduced to play the part of a specimen value and the confusion is removed, but there seems no point in making it *just to imitate a common practice in mathematics* (where one says, for example, let n be the least integer such that . . .).

The theory of classes which Rosser develops is Quine's Zig-Zag theory, which replaces the Russell-Whitehead theory of types by the method of potential typing known as stratification. Instead of assigning suffixes to all variables to separate them into types, variables of one type forming elements of classes denoted by variables of the next higher type, as in type theory, in the Zig-Zag theory variables are all kept on the same level and a formula is said to be stratified if it is *possible* to assign suffixes to the variables so that the type condition is fulfilled. Thus

$$(y)\,(x \in y \,.\supset (x)\,(y \in z \,.\equiv .\, x \in y))$$

is stratified because we may write it in the form

$$(x_1)\,(x_0 \epsilon x_1 \supset (x_3)\,(x_3 \epsilon y_3 \,.\, \equiv \,.\, x_0 \epsilon x_1))$$

where each occurrence of the membership connective ϵ is flanked by variables whose suffixes increase by unity from left to right. The class of stratified statements is greater than the class of statements allowed by type theory, since clearly any statement which satisfies the theory of types is stratified, but whereas both $x \epsilon y$ and $y \epsilon x$ are stratified only one of these two statements could satisfy type theory (since one would be $x_0 \epsilon x_1$ and the other $x_1 \epsilon x_0$). Stratification is used as a (permissive) condition for a statement to determine a class, so that the Russell paradox is excluded, not by prohibiting certain statements but by requiring only stratified conditions to produce classes.

In the chapter on " Relations and functions " Rosser stresses the importance of maintaining a distinction between a " function " f and its " value " $f(x)$. By means of Church's lambda notation the " function of x whose value is x^2 " is denoted by $\lambda x\,(x^2)$; thus for instance the relation between the exponential function and its derivative is expressed by

$$D\,(\lambda x\,(e^x)) = \lambda x\,(e^x).$$

While it is clear that the distinction drawn is a clear and useful one, already recognised in analysis when we denote the derivative of $f(x)$ by $f'(x)$, emphasising that the relation is between f and f', Rosser's proposal to reserve the term function for the " f " and to call " $f(x)$ " the value of f for x, though reasonable enough, is awkward in use. It seems preferable to continue to call " $f(x)$ " the function and perhaps to call " f " the operator, so that a function is the application of an operator to a variable and a function value the result of applying the operator to a value of the variable. The drawbacks of Church's lambda notation are too obvious to need stressing. The aim of all good notation is to reduce the number of superfluous variables, whereas the lambda notation turns e^x into $\lambda x\,(e^x)$. The need which Church sought to meet in this way is probably better met (in mathematical writings) in the traditional way of introducing new function signs, as for instance when we introduce say $F(x)$ to denote $\exp\{1/\sqrt{(\log x)}\}$. In logic of course the needs are different and we should distinguish between a notation which is good for our purpose (for example, keeping distinct things different) and one which is good for practical purposes (like Leibnitz's dy/dx).

Rosser makes the now common mistake of supposing that because definition by recursion is not *eo ipso* a function definition in the theory of relations, it is therefore fallacious to suppose that a function may be defined by recursion. It is simply a question of the nature of the system in which the definition is made. In a system which admits the equations

$$x + 0 = x,\ x + Sy = S\,(x + y),\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

(where Sx is the successor of x) *as axioms* the sum function is thereby as well defined as one could wish. However, in the theory of relations it is possible to define *explicitly* a sum function such that the recursive equations (i) become provable. Rosser gives a new and interesting proof of this both for finite and transfinite recursions, (he is wrong, however, in attributing to Landau the observation that one cannot justify definition by recursion until one has some of the properties of an inequality relation. Landau, in fact, says (*Grundlagen der Analysis*, 1948, p. x) that Kalmer did justify the recursive definition of addition without introducing $x \leqslant y$).

The principle of mathematical induction is proved only for stratified statements. A consequence of this is that it is not possible to prove in the formal-

ism that the class of m's satisfying $0 < m \leqslant n$ has n members. Each particular instance of this result, with $n = 0, 1, 2, 3, \ldots$ is provable, but so far as is known the general case with a variable n is not provable. That is not to say that the general result has been proved to be non-demonstrable, only that a method of proof is lacking. To meet this deficiency Rosser introduces a counting axiom which ensures that the class of m's for which $0 < m \leqslant n$ has n members. The need for this axiom is particularly interesting because it was always thought to be one of the triumphs of Russell's derivation of mathematics from logic that counting was definable in terms of the similarity relation.

The last chapter of the book is devoted to the axiom of choice and the proof of the equivalence of eight forms of the axiom (three due to Zermelo and one to Zorn). Rosser suggests that in elementary analysis the axiom is never needed (though sometimes tacitly used), but that the Lebesgue theory of measure requires the denumerable axiom of choice and the construction of non-measurable sets perhaps an axiom of choice of higher cardinal.

Logic for mathematicians is perhaps the only modern book on logic which does not make any use of Gödel's work on the arithmetisation of syntax and the construction of a non-denumerable proposition, nor any reference to the consistency of the continuum hypothesis. Presumably these are thought to be topics of less interest to the mathematician than to the logician and if this is the reason then Rosser's restraint in omitting an account of the Gödel theorems is all the more remarkable in that Rosser himself discovered the best form of one of the theorems. The book has however many more solid claims to distinction than the omission of the Gödel theorems.

R. L. GOODSTEIN.

Calculus. By C. R. WYLIE, Jr. Pp. x, 565. 48s. 1953. (McGraw-Hill)

There are many universities in the U.S.A. and from several of them in recent years have come books on calculus and differential equations. These books have features in common which distinguish them from English text-books. They are intended for students who have not met calculus at school and are perhaps entering on a university course after a gap in their studies, for the authors betray an uneasiness that the elements of algebra, trigonometry and coordinate geometry may not be remembered. Nevertheless, since the books are for university students, more attention is paid to rigour than is usual in English text-books written for a first course at school or for technical students. It is probable that many of our graduates in science and engineering, who now-a-days acquire a very respectable degree of skill in mathematics, have never had clear ideas on limits and continuity, convergence, or the validity of first order approximations used in forming integrals. Such graduates might find it useful to revise their calculus by reading such a text-book as this, though many of the exercises will be to them trivial.

Professor Wylie's book starts with a chapter entitled "The Calculus in perspective" introducing the two concepts of slope and area with examples of their physical significance. After a chapter on limits and continuity the usual subject matter is dealt with in sizable portions. Formal differentiation of algebraic, trigonometric, logarithmic and exponential functions is covered in one chapter, followed by applications, including curvature in Cartesian coordinates and the mean value theorem, in another. Integration is defined as summation; with methods and applications it occupies Chaps. 5 to 10. Hyperbolic functions are introduced in Chap. 11 by means of the solution of certain problems in terms of logarithmic and exponential functions, so that the convenience of the new functions and the analogy with circular functions is apparent. Quite good chapters follow on partial differentiation, convergence,

expansion in series (including Fourier series), multiple integrals and differential equations. These last include separable, homogeneous and linear equations of the first order and equations of the second order with constant coefficients.

As the author states in his preface, there is rather more written matter in the form of preliminary discussion of a topic, explanation of the steps of worked examples, etc., than is often given in text-books. This feature should be an advantage to the reader who is studying without the help of a teacher. One feels a little envy at such lavish use of that expensive commodity, paper.

C. G. P.

Quadratische Formen und orthogonale Gruppen. By M. EICHLER. Pp. xii, 220. DM 24.60; geb. DM 27.60. 1952. Grundlehren der mathematischen Wisenschaften, 63. (Springer, Berlin)

The merest glance at this book shows that the theory of numbers still continues to grow at an enormous rate. Not only are new points of view being developed but new ideas and methods are being applied to old results. These are then re-interpreted and presented in so different a manner that the old results become almost unrecognizable, and often because they have become embodied in a more extensive and general setting. To understand the new approach, which may involve a close connection with other subjects that may have once seemed remote from the present study, more and more advanced knowledge of modern algebra and other parts of number theory are required. This, of course, does not conduce to easy reading, and so a reader may easily feel lost without an intimate knowledge of what is required for a fuller comprehension. There is, however, another and very important aspect of all this. The new points of view often produce a great gain in unity and in simplicity, conciseness and elegance of presentation. All this is confirmed by the present book, which gives an account of the most recent developments in the theory of quadratic forms.

From the theory of the binary quadratic form as developed by Gauss, two further studies arose. One was the theory of quadratic forms in n variables, and the other, the arithmetic of algebraic number fields and that of hypercomplex number systems. The second study had attracted greater attention for some time past because it became possible to reduce the investigations on algebraic fields and algebras to a central theme, developing them from elementary foundations. But now the recent work of Hasse, Hecke and Siegel on the theory of quadratic forms has made it possible to develop the general theory afresh.

The author proposes to make the new ideas available to wider classes of readers and so has to reconsider the subject from its beginning. He considers the theory of quadratic forms as a geometry in a vector space over an arbitrary field, and in particular an algebraic number field, endowed with a metric comparable with the usual Euclidean one. He alternates investigations on the quadratic form defining the metric with those on the general orthogonal group for which the form is invariant. The great difference between the old approach and the new is evident at once from the headings of the chapters. These are:

(1) Algebra of the metric spaces,
(2) Metric spaces over fields with a perfect discrete valuation,
(3) The elementary arithmetic of metric spaces over algebraic number fields and function fields,
(4) Vectors and ideals,
(5) The higher arithmetic of metric spaces and in particular over the field of the rational numbers.

The titles give perhaps little indication of the results on quadratic forms contained in the chapters, but these contain a great deal of substance from the theory of quadratic forms, in fact, some of the deepest and most important results. Thus the fifth section contains an account of the mass or weight of quadratic forms, and also of their automorphs; and section four contains an account of the transformation of the general theta functions under the substitutions of the general modular group. The book, however, is by no means a handbook. Neither is it exhaustive; thus it contains no account of the reduction theory of quadratic forms.

Sufficient has been said to show that the book is indispensable not only for those interested in the arithmetic of quadratic forms but also for those interested in the many developments which have resulted from the arithmetical theory. There are not many books which contain in so small a compass such a wealth of ideas and material.

L. J. MORDELL.

A Textbook of Matrices. By SHANTI NARAYAN. Pp. vii, 289. 7 rupees 8 annas. 1953. (Chand, Delhi)

The preface to this book states that " no pains have been spared to make the treatment lucid, even though it has been at places possible only at the cost of brevity ". This is also the reviewer's opinion of this excellent introduction to the theory of matrices, which deals with fundamental concepts, the algebra of matrices, the rank of a matrix, vector spaces, linear equations, real and Hermitian quadratic forms, equivalence, congruence, and the simpler cases of similarity. The author however assumes on the part of the reader a knowledge of the theory of determinants and the book might not suit those who prefer to introduce a determinant as a function of a square matrix. It is a pity also that room was not found for a more comprehensive account of similarity. The printing is in places clumsy and occasionally erratic, though seldom misleading. In Ex. 6 on p. 245 the reader is asked to prove that if A is a real orthogonal matrix then either $A - I$ or $A + I$ is singular, whereas Ex. 7 belies the statement. The result is, of course, true if A is of odd order.

In spite of some minor blemishes this textbook will be welcomed by students seeking a sound but elementary exposition of matrix theory.

D. E. RUTHERFORD.

Mémoires sur la Théorie des systèmes des Équations différentielles linéaires. By J. A. LAPPO-DANILEVSKY. Pp. viii, 253, 208, 204. Bound in one volume. $10. 1953. (Chelsea Co., New York)

In these remarkable memoirs, which are a collection of work published during the author's lifetime, the late J. A. Lappo-Danilevsky first combines the methods of matrix algebra and the theory of analytic functions, in order to develop, to an extent sufficient for his purpose in this work, the theory of functions of a matrix variable, and of matrix functions of a complex variable. This done, he then applies this technique to the theory of systems of linear, ordinary differential equations, which he develops in very great generality, and most elegantly.

At first sight, the book seems to be very difficult reading, owing to the special notations required, and the amount of preliminary theory which precedes the main objectives. But in fact, these notations and theory are only extensions of the matrix methods with which the modern applied mathematician and theoretical physicist are familiar; and to master them will present no difficulty to anyone familiar with the vector, tensor and matrix methods which have revolutionised modern mechanics. And indeed, the parallel is a very close one. For vector, tensor and matrix methods, though in themselves

containing no fundamental theoretical advance, have by their inherent conciseness enabled work to be done which would be quite prohibitively " heavy " if written in the old, full notation. And in exactly the same way, if the method of Lappo-Danilevsky be adopted by those who work in branches of modern theoretical physics involving systems of linear, ordinary differential equations, with a large number of dependent variables and equations, their very great conciseness may well lead to spectacular advances.

It is remarkable, and deplorable, that Lappo-Danilevsky's work seems to have passed almost unnoticed, although the first of these three volumes of memoirs was first published in 1933. The Chelsea Publishing Company are to be warmly congratulated upon their vision in bringing out the present edition, comprising the three volumes bound as one, of a very significant book.

D. H. P.

Éléments de Mathématiques. By N. BOURBAKI. XIII. Livre VI. *Intégration*. Pp. 237. 2500 fr. 1952. (Actualités scientifiques et industrielles, 1175 ; Hermann, Paris)

The present volume contains the first four chapters of the book on Integration in the encyclopædic series of Bourbaki. It brings a point of view about Integration which, if not absolutely novel, is certainly set out here for the first time in any memoir on the subject. The presentation has been the result of considerable thought, and, according to those who know the eminent author, has emerged from the consideration of a number of draft versions of different points of view on the subject ; the result is of great interest.

The first point is that the integral—Lebesgue—or Lebesgue-Stieltjes—is considered as a functional, defined on a class of functions : it is not introduced via the notion of measure : on the contrary measure is defined as a special case of an integral. For the central part of the book the class of functions considered consists of the continuous functions on a locally compact space. The class of continuous functions is a vector space and it would be possible to define the integral as a functional on this space by considering it as a topological vector space ; but the method followed is different : it is essentially a development of that due to Daniell, and is one which lays emphasis on the ordering of the continuous functions, rather than on their convergence properties ; and it is justified by the great importance of relations of order of functions in the theory of integration. The functionals concerned are built up out of positive functionals, those which are defined for all continuous functions vanishing outside a compact set.

To build up the theory from this basis a theory of ordered vector spaces is needed : and the necessary theory, that of vector lattices, or, as Bourbaki calls them, " Espaces de Riesz ", is developed in the second chapter. The third chapter then develops the theory of measures, defined in this manner as functionals.

The fourth chapter defines integrals of the form $\int f\,d\mu$, where μ is a measure on a locally compact space and f a function defined on that space : the extension of the notion of integral to continuous functions with values in a Banach space, and to such functions belonging to an L^p class, is treated in detail. Incidentally, it is shown that it is possible without undue complication to distinguish between the L^p spaces and the corresponding spaces of functions with summable pth power. The first chapter of the book is concerned with inequalities of convexity, and contains interesting complements to the general theorems in the famous book of Hardy, Littlewood and Pólya. This subject is taken up again in the fourth chapter, where there is also to be found a rather

brief discussion of abstract measures, defined for sets which have not necessarily a topology; these measures are also defined as functionals on a Riesz space of functions defined on the abstract set, a method suggested independently by Stone and Bourbaki.

As an exposition of the theory of measure this book departs widely from existing ones. One of its most debatable aspects is the concentration upon measures defined on locally compact spaces. This is justified by two arguments: firstly that it is fulfilled in all the applications of the theory, and secondly that in any case it is always possible to give a set on which a measure is defined a topology which brings that measure within the theory. The second is not a strong argument: for it is based on a construction which defines a very complicated topology even in simple cases. The first argument too is one which would be contested by workers in probability theory and related fields. Nevertheless, it holds good for all the ordinary applications of analysis; and the point of view of this book is the one which is most illuminating for the study of many modern developments of analysis.

The equipment required by the reader of this book is decidedly heavier than that needed by other expositions of Integration theory; it is, indeed, difficult to believe that anyone with a normal mathematical education would be able to acquire the knowledge of the theory of linear spaces necessary for the reading of this book without having learnt at any rate the theory of Lebesgue integration for the real variable; the only escape from this circular route must wait upon the publication of Bourbaki's own work on these spaces. However, anyone with a knowledge of the first few chapters of Bourbaki's topology and of the elements of the theory of Banach spaces should find little difficulty and a great deal of instruction in this book, not only from the text but also from the well selected examples which give further insight into the theory.

J. L. B. Cooper.

Einführung in die höhere Mathematik. By Georg Feigl and Hans Rohrbach. Pp. viii, 375. DM 26.80. 1953. (Springer, Berlin-Göttingen-Heidelberg)

This book is the printed version of a course of lectures which between the two wars was a regular feature of the study of Mathematics at the University of Berlin, and to my knowledge was peculiar to Berlin. Through this course Georg Feigl introduced generation after generation of future mathematicians, physicists, chemists, and engineers to University Mathematics. The lectures were the result of much thought and planning—Erhard Schmidt's influence can be felt in many places; it was Georg Feigl's gift as a teacher and his enthusiasm for the course which kept it fresh year after year. The reviewer was one of the hundreds of students for whom it was the first taste of Mathematics proper and remained one of the lasting impressions of her student years. Re-reading it after many years, including some years spent in University teaching, she still finds the "Einführung" an admirable course of lectures.

Georg Feigl died in 1945 and the lectures were edited for publication by Hans Rohrbach. Himself brought up on the course, later giving it in Georg Feigl's place when the latter left Berlin, he has entirely preserved its essential character in spite of those unavoidable changes implied by the difference between a book and a course of live lectures.

The aim of these lectures was twofold: their contents was to provide some of the tools used in various branches of Mathematics and Physics; their form was designed to bridge the gap between school and university. This gap is familiar here, but was much wider in a country where no specialisation was allowed in schools, where on the other hand the nature of undergraduate lectures was such as to produce, where successful, students well able to find

their way about the original literature, imbued with the spirit of research, but —at least in one case—unable to perform concrete integrations or to solve differential equations. Thus consideration for the beginner and a high level of abstract thought applied to a considerable range of material are the characteristic features of the book.

The ground covered is as follows. Elementary vector algebra in n dimensions, determinants, polynomials and rational functions, a complete treatment of systems of linear equations, the elements of group theory, and matrix algebra provide tools for geometers, algebraists, and physicists. Next the ground is prepared for the analysts: A chapter on theory of sets including ordered and well-ordered sets leads on to the final part (nearly a third) of the book which gives a full axiomatic treatment of the number system. Starting with the natural numbers whose properties are derived from a set of axioms (Erhard Schmidt's) equivalent to Peano's, the system of integers, then that of rationals, are constructed step by step. Finally the reals are introduced as segments ("Anfangsstücke") of the ordered set of rationals and their properties derived. The proof of the continuity of the set of reals in terms of Dedekind sections and its order isomorphism with the set of points on the line concludes the book.

The presentation takes meticulous care over detail, every statement is proved in full, simple numerical examples are given wherever possible. There are liberal explanations of the aims and reasons of a procedure, and the nicer logical points are put clearly. Thus e.g. the idea of a polynomial as a function of a variable, the point of view adopted in Chapter IV, is compared at the end of the chapter with the concept of a polynomial in an indeterminate as it occurs in abstract algebra. Nothing is taken for granted: the "if and only if" with its two part proof is fully discussed where it first occurs; a whole paragraph in the first chapter is given over to the rules of handling summation signs. Here double sums give opportunity to introduce the double suffix notation, and matrices are mentioned for the first time, to be used gradually more and more, until they are fully treated in their own right in Chapter VII. Determinant theory is developed for orders two and three before the general case is attacked.

Two more examples will suffice to indicate the didactic method and the scope of the book. Chapter II on vectors begins with the calculus of n-tuples of numbers, in particular their linear dependence and independence over a number field. Then a linear space over a field is defined in abstracto, the notions of basis and dimension are introduced, and illustrated by the space of n-tuples of numbers. Scalar product and orthogonal systems follow, and the Cauchy-Schwarz inequalities find their place here by way of application. In the next paragraph we see vectors in one to three dimensions introduced geometrically, in relation to their physical applications. Cartesian co-ordinate systems are explained, and the close relation between operations with vectors and the calculus of real *n-tuples* for $n = 1, 2, 3$ is established. This in turn is used in the last paragraph to define the n-dimensional number space, both real and complex, and so to interpret the algebra of n-tuples geometrically as vector algebra. An application of determinant theory finally yields Erhard Schmidt's orthogonalisation.

Specific examples of equivalence relations and mappings occur in the chapters on groups, matrices, sets and ordered sets. At the end of the chapter on set theory the abstract concepts of equivalence relation, and of isomorphism between sets with binary operations are treated. The reader is now equipped to follow the standard process of embedding a commutative semigroup in a group in the two concrete cases where the semigroup is the additive semigroup of natural numbers, then the multiplicative semigroup of integers.

This process of obtaining the rationals and their properties is carried through right up to the point where all the usual rules on handling equalities and inequalities emerge. E.g. on p. 319, the associative law of addition is explicitly shown to imply the rules $a+(b-c)=(a+b)-c$, $a-(b+c)=(a-b)-c$, and $a-(b-c)=(a-b)+c$; their multiplicative counterparts lead to the rule: "To divide by a fraction, turn it upside down and multiply,"—thus showing the student that what he has done unthinkingly at school must be, and can be, justified. Or again, on p. 321, the frequent need to obtain estimates, and hence to derive inequalities from equalities, is explained; corresponding rules, such as "$ab=c$ and $a>0$, $b>1$ implies $a<c$" are established, and Bernoulli's inequalities follow as illustrations.

The book contains no problems. And no amount of careful study will help a student to "do examples". It does however lead the open-minded beginner a long way towards a deeper understanding of the spirit of Mathematics, quite apart from providing him with a solid store of knowledge to help him on his way. It is a matter of deep regret to the reviewer that a course of this kind hardly fits into the curriculum of our universities, where so much more time is spent on enabling a student to solve problems—or perhaps: so much more care is taken to turn out students not worried by an integral or a differential equation.

HANNA NEUMANN.

Analytische Geometrie. By G. PICKERT. Pp. x, 397. DM. 26. 1953 (Akademische Verlagsgesellschaft, Leipzig)

If the quantities used in analytical geometry are elements of a skew (not necessarily commutative) field, much greater generality is obtained, and the work involved is not much more difficult. This realisation is the inspiration of Dr. Pickert's admirable book. It is hardly true to say, however, that this idea has been absent in previous books on analytical geometry. Segre's *Lezioni di geometria moderna* has the same *leit-motif* running through it, and so has *Methods of algebraic geometry*, vol. 1, by Hodge and Pedoe, which appeared at the same time. The author of the book under review refers to the first book, but not to the second. The only justification for a new book would be that it is in a different language, or that it covers different ground, or both. It can be said that the new arrival does cover some aspects of elementary geometry in more detail than either of the two prior books. For instance, after collineations in n-space have been discussed, space is devoted to the cases $n=2$ and $n=3$. This descent to the more familiar is one of the pleasing features.

The three main sections are entitled "Affine geometry", "Metrical geometry" and "Projective geometry", and are thorough. Necessary algebraic theorems are proved in the text. One curious omission is the theory of elementary divisors. As a consequence collineations are only classified in some special cases. But as the book is evidently intended to be an elementary text-book on analytical geometry, with the necessary n-dimensional algebra put in, this is not a very serious defect.

There is far more detail than in the early chapters of van der Waerden's *Einfuehrung in die algebraische Geometrie*, and a student reading van der Waerden, and having faith in German texts, could usefully dip into the Pickert book. The diagrams are good, and there is an index.

D. PEDOE.

Lectures on Analytic and Projective Geometry. By DIRK J. STRUIK. Pp. ix, 291. 1953. (Addison-Wesley, Cambridge, Mass.)

This very readable and attractive book is based on a one-term course of lectures to American students at the Massachusetts Institute of Technology who have already passed through a three-term course of analytical geometry and calculus, and have some familiarity with cartesian co-ordinates applied

to curves in the plane, can recognise a quadric surface, and know some vector algebra.

At first glance the text is of a type very familiar to English readers, with its mixture of analytic and synthetic methods, but in many ways it goes much further than elementary textbooks of the Sommerville *Analytical Conics* type. On the other hand it is not as thorough or as self-contained as Todd's book *Projective and Analytic Geometry*, and although the many and detailed references are a pleasing feature of the book, the student may sometimes be perturbed at the amount of information he is expected to find elsewhere. I well remember a lecture given to the Trinity Mathematical Society in which the lecturer began by covering the blackboard with more than fifty references to papers on his topic, a certain finite group. The majority of his audience took a mental oath never to add to the list at any time! The references in Prof. Struik's book may possibly arouse a similar thought in the minds of some readers.

But this is a very minor blemish. There are many features which call for special notice, as similar English textbooks do not mention them. For instance, there is a section on orientable and non-orientable surfaces, with a photograph of a Klein bottle which is evidently made of glass ; a section on groups and the Erlangen programme ; a chapter entitled " Projective Metric ", which discusses non-euclidean geometries. Two chapters on quadrics are followed by one on space transformations, with sections on the various types of projection, including orthographic projection, or descriptive geometry. The book ends with the projective theory of null systems, affine and euclidean theory, forces and force systems.

Each section has an ample number of carefully selected examples. The answers at the back provide hints for solution. There are also additional exercises and an index. The diagrams are excellent. There is a reproduction of Duerer's " St. Jerome " to illustrate the theory of perspective, and a section on how to make good diagrams of solid figures.

After working through this book a student would know whether he was attracted to geometry as a mathematical discipline or not. His background of general mathematical culture would have had much added to it. This book should be in all sixth-form libraries, and is recommended to all university students.

D. PEDOE.

Co-ordinate Geometry. By C. O. TUCKEY and W. ARMISTEAD. Pp. xi, 464. 12s. 6d. 1953. (Longmans)

Co-ordinate Geometry by C. O. Tuckey and W. Armistead, follows closely, as one would expect, the recommendations contained in the Association's recently published Report on the Teaching of Higher Geometry in Schools, since the authors are members of the sub-committee which prepared that report. The first four chapters, on Points, Lines, Curves, and the Central Conics supply all that is required for *A* level, and the whole book should prove adequate for *S* level.

The parabola is introduced as the curve of squares, the hyperbola as the curve of reciprocals, and the facility of these definitions inspires the consideration of the cubic, the trigonometrical and the exponential curves. Curves of these last two types are by their nature essentially arithmetical, and their properties can not be stated in terms of location, which is the essence of Geometry, without the assistance of calculation, which is the essence of Arithmetic. They have their place, of course, but they form no part of geometry. They give, too, to the equation $y=f(x)$ for a curve an importance which should be reserved for $f(x, y)=0$, in accordance with the ideas of symmetry so commendably urged in the rest of the book.

Room might have been found for a discussion of the line-pair through the origin in greater detail than the few lines allotted to it. The purist may be offended by : (p. 5) "If $P(x, y)$ is any point on the circle whose centre is at the origin and radius is a, then $x^2+y^2=a^2$ is the equation of the circle" ; (*ibid*) "In each of these cases the terms of the second degree in x and y are the same, viz. x^2+y^2" ; and (p. 42) the use of the word "coefficient" instead of "constant". A reader, too, may waste a few minutes in trying to establish the congruence attributed to the triangles *LMS* and *LMN* on p. 388 before discovering that they are not congruent.

But these are minor blemishes in what is normally a very clear and forthright style. The bookwork is well presented, and the exercises numerous and stimulating. The pupil who masters the book, or the appropriate part, will be well equipped for the task that lies ahead of him. W. J. HODGETTS.

La Géométrie et le Problème de l'Espace. Tome V : Les Géométries non Euclidiennes. By F. GONSETH. Pp. 110. 17s. 1953. (Griffon, Neuchatel)

Consider the interior of a fixed circle C. This is to be our space. In this space lines are defined as the portions within C of those circles which cut C orthogonally. It is easily seen that through two points in C only one line can be drawn. Two lines either meet in a point or not at all. Two lines which do not meet are said to be ultra-parallel. Except for the parallel axiom, all Euclid's axioms hold for the points and lines in our space. Through a given point of the space an infinite number of ultra-parallels can be drawn to any given line. Distance can be defined, and a metric set up for which the boundary of C becomes inaccessible, or "at infinity". Shakespeare was evidently thinking of this model of hyperbolic non-euclidean geometry when he makes Hamlet say "I could be bounded in a nutshell and count myself a king of infinite space. . . !"

Prof. Gonseth explores this fascinating space in detail, and his book, like all good guide-books, makes us eager to explore the terrain for ourselves. Any good Sixth-former would have no difficulty in understanding the text, and it should be required reading for all first-year University mathematicians who, it must be admitted, often lose their zest for geometry when confronted with an overwhelming syllabus and insufficient time for its proper mastication and digestion. The book under review would act as a tonic for flagging appetite.

I was stimulated to look through a number of elementary textbooks on geometry to see whether any indication of the exciting things outside the scope of these books is given. In many of the popular geometry books of today the examples have been so carefully sifted that they are now emasculated. I did find one book which shows what an exciting subject geometry can be, and actually ends with a broad hint that there is something beyond Euclidean geometry. This book, *Modern Elementary Geometry*, by L. Roth (Nelson, 1948, 8s. 6d.), was severely criticised in the *Gazette* when it appeared, partly, it seems to me, on the grounds that it is an impertinence for University mathematicians to write books for schoolboys. But surely this practice, which is universal on the Continent and in the States, is to be commended if fine books are produced? Prof. Gonseth's books are deliberately aimed at a larger public than that constituted by University students, and can be recommended to all who wish to have a glimpse of the beauties of geometry. D. PEDOE.

Introduction to the Theory of Statistics. By V. GOEDICKE. Pp. ix, 286. 35s. 1953. (Harper, New York ; Hamish Hamilton, London)

This is a book in the Harper's Mathematics Series, written by the Associate Professor of Mathematics of Ohio University. He suggests that although the

mastery of statistics for most students is almost entirely " operational " yet it is possible and desirable to give such beginning students a deeper understanding of theory than is customarily attempted. This book is therefore written for such students.

It consists of 13 chapters, of which three are, in the main, about usual school algebra : there is no calculus in the treatment at all, and negative logarithms are dealt with by the old -10 method. Some of the treatment is, in fact, rather too non-rigorous, as in the use of " exactly " on p. 5, of " somewhat " on p. 278, and of " confirms " on p. 205. But on the whole, it is a sound development of the subject up to multiple and partial correlation and chi-squared (which the writer so calls on p. 226, though elsewhere he follows the American practice of calling it chi-square). The data " are fictitious unless otherwise noted ". Among the former are some based on " dating " ; among the latter are a number from Croxton and Cowden (see *Math Gazette*, XXVI, No. 271 (October, 1942), p. 194), including some on the brain weights of U.S.A. Senators, as well as some on the change in attitude of Judges of the Supreme Court as shown by their voting record. It is not clear if the " Galaxies " question is intended to be a fictitious one.

The book is attractively got up, though some of the type used for suffixes is very small and not too easy to read ; in at least one case (on the top of p. 161) it has led to a misprint (y for z). The distinction made, by the type used, between $\bar{x}^3$ and $\overline{x^3}$ is so very slight that when we come to deal with terms such as $(x - \bar{x})$ similarly treated with the double-deck bars the appearance is very confusing. It may also be confusing to find a frequency curve given on p. 273 as the answer to a question on p. 32 demanding a histogram and to find the open circles of the graph on p. 181 described on p. 180 as crosses. It is a pity that the tables of numbers such as are given on pp. 171, 200 and 204 are printed without rules or spaces between, say, groups of 5 ; this sort of thing is one of the faults that used to appear in the cautions in the old books on descriptive statistics.

Each chapter is divided into articles, with problems at the end of each article (when appropriate), the tables and graphs being referred to by the article reference number, and the chapter concluding with a summary of its contents and argument. At the end of the text we find (p. 261) an appendix of six tables (one of them the American Experience Mortality Table of 1920), a table of formulas (including the algebraic rules of indices), a " selected list of books ", (the great majority of the thirty being American ones), answers to the problems that appear at the end of the various articles of the chapters, with finally a two-page index.

Among the various chapters, the best are, I think, those on the Normal Curve (Ch. 6) and on Correlation (Chs. 9 and 12). I was glad to see a reference to the Lorentz Curve again. The chapters on sampling are not so good : they are marred by fáiling to give a clear warning about the distinction between the s.d. of the population and that of the sample available : a further point about the chapters on this topic is that " confidence limits " are needed on p. 195 though they are not defined until p. 217.

In all, it is an interesting treatment of the topic that may be worth examination by teachers, though the price will hardly make it a suitable text for general use.

FRANK SANDON.

Stochastic Processes. By J. L. DOOB. Pp. vii, 654. 80s. 1953. (New York : Wiley. London : Chapman and Hall)

In 1826 the English botanist Brown observed through a microscope the irregular and apparently spontaneous movements of grains of pollen suspended

in a liquid. This irregular movement, now known as the Brownian motion, was later attributed to the impacts of the molecules of the liquid on the grains of pollen. It provided circumstantial evidence in favour of the kinetic theories of matter, and it also provided applied mathematicians with a new type of problem. In the case of the Brownian motion it was to predict, in terms of probability, the position of the particle after a time interval large compared with the mean time interval between successive molecular impacts ; a solution of the problem of the Brownian motion was given by Einstein in 1906. Many other physical processes have since been met in which the variables observed change erratically though controlled by probabilistic laws, *e.g.* the occurrence of " noise " in physical systems emitting very weak signals, diffusion processes, the dispersal of crowds or the movement of traffic. Many practical problems arose out of such processes during the last war, and they were rather unsystemically solved in various ways. Since the war it has been possible to review the methods of solution and to explore the mathematical ideas which underlie them. Most of the systematic exploration of stochastic processes, which are the mathematical models of random physical processes, has been carried out in recent years by American mathematicians, prominent among whom is Professor J. L. Doob, the author of the book under review, who is already well-known in this country through his original and important contributions to the Berkeley Symposia on Mathematical Statistics and Probability.

Professor Doob develops and systematizes the mathematical theory of stochastic processes on the basis of Kolmogorov's work in establishing probability theory as a branch of pure mathematics. His approach is uncompromisingly mathematical—he is Professor of Mathematics at the University of Illinois. Though he uses in part the terminology of stochastic processes as the subject was developed in terms of classical probability theory, and thereby helps the statistical reader intuitively, that is the only concession he, as a pure mathematician, makes to his more practical readers. He defines a stochastic process as any family (usually infinite) of related random variables $\{x_t, t \in T\}$ where, in practical terms, x_t is the observation at time t within the time range T. In this book the author restricts himself to families in which T is a linear point set and the random variables are almost always real or complex numbers. His main tasks are then to explore the properties of such families, to classify them, and to devise new types of stochastic processes which may be useful, or are mathematically elegant, or are interesting in other ways. He finds plenty of scope both for his ingenuity and for his systematizing ability.

The author accepts from the beginning that probability theory is a branch of measure theory, and thus many readers in this country who may be interested in the practical applications of stochastic processes will find it difficult to read even the first chapter. To help such readers the author adds as a supplement a summary of the main theorems of measure theory which he applies in his main text, but few will find this supplement adequate as an introduction to measure theory. The author begins by showing how statements made in probability language can be interpreted in terms of the language of measure theory ; every function of random variables is shown to have a representation whose basic space is function space. He then defines a stochastic process in the form already given and immediately classifies the main types he proposes to study. They are those with mutually independent random variables ; those with mutually uncorrelated or orthogonal random variables, discrete and continuous ; Markov processes ; martingales ; those with independent or with orthogonal increments ; and discrete and continuous stationary processes. He concludes with a chapter on linear least squares prediction theory. It will be seen that the book is comprehensive in scope. Moreover it brings together for the first time in a coherent form the subject

matter of many papers published in many languages in recent years, and converts this scattered material into an organized and systematic mathematical unity.

It must be emphasised that this is the first *mathematical* book to be written on stochastic processes : it is therefore to be welcomed as evidence of growing mathematical interest in probability theory. Unfortunately there seems to be in Britain a complete dichotomy between pure mathematics and statistics, perhaps because statistics here arose from practical problems rather than from mathematical theory. Whatever the cause of this dichotomy may be, it is becoming clear that Britain is lagging behind in the field of mathematical probability theory ; elsewhere, and especially on the Continent and in the United States, probability theory has long attracted the interest of pure mathematicians. The situation is made apparent in the nine pages of references to original papers published over the last fifty years which is given at the end of this book ; not a single British author or paper is mentioned. This book may therefore meet a special need in Britain by demonstrating that probability theory is elsewhere now fully established as a branch of pure mathematics.

In format and printing the book maintains the very high standard set by the Wiley Publications in Statistics. Though the author's treatment of his subject is formal and rigorous, his style is informal and readable. The structure of the main exposition is made clear in the first two chapters, and thereafter the reader, guided by helpful notes and frequent back references, always knows where he stands in relation to the main theme. It is a book which not only marks an important stage in the consolidation of a hitherto rather rarefied subject but it will long be used as an indispensable work of reference by all interested in stochastic processes. In this country its greatest service may be to interest pure mathematicians in a subject they have too long neglected.

B. C. BROOKES.

Cambridge Elementary Statistical Tables. By D. V. LINDLEY and J. C. P. MILLER. Pp. 35. 5s. 1953. (Cambridge University Press)

A set of tables selected for elementary work from the larger and much more expensive standard tables, such as the *Tables for Statisticians and Biometricians* and others published by *Biometrika*, has long been needed by those who teach or use elementary statistical methods. Happily the long-awaited set is authoritative and meets all reasonable requirements ; the selection made, the attention given to details, the printing and lay-out are all excellent.

The statistical tables included are of :—The Normal distribution and frequency function ; percentage points of the Normal, the t-, the χ^2-, and the F-distributions ; Fisher's z-transformation of the correlation coefficient ; factors for converting mean range to standard deviation ; and random sampling numbers (4000 digits from the tables of Kendall and Babington-Smith). In addition there are 22 pages of tables of squares, square roots, reciprocals, logarithms, and logarithms of factorials for use in computation.

No single publication could do more to encourage the teaching and application of statistical methods (advanced as well as elementary) than this new set of Cambridge Elementary Statistical Tables.

B. C. BROOKES.

A Background of Primary School Mathematics. By L. D. ADAMS. Pp. xv, 189. 8s. 6d. 1953. (Oxford University Press)

This is a book based on observation of children and their learning of mathematics. Its main contention is that such learning can only take place against the background of the child's whole experience, and that this is necessarily a different background for each child. In the early stages, the experience is

all important, and no re-iteration of words or symbols can take its place: in fact, according to the writer, if such words and symbols are introduced too early, and not adequately related to experience, they hinder, and may even prevent, mathematical development. An able child learns more quickly from his experience, and each bit of it therefore recedes more quickly into the background of his thinking, than is the case with a slow child; but a rich experience is necessary for all. Miss Adams tells stories of some eighty individual children, of all ages from two to twelve, observed in their homes, in school, or in chance meeting in train or 'bus; she describes work seen in progress in about fifteen classes and she suggests other ways in which primary school mathematics may be linked with the children's experience. The teachers and children referred to are incognito, and her pictures are composite ones, but she has combined them to make a readable account from which teachers will learn much. For the book is addressed to teachers, and intending teachers, and, though some will find in it implied criticism of methods they have practised, all will appreciate the sympathetic insight into situations as they confront the learner. Some parents, and, perhaps, some teachers, will be surprised to discover how much more there is in learning mathematics than the rote mastery of tables, and the careful carrying out of " rules " shown by the teacher. Perhaps some reader will feel constrained to examine anew the background of his own mathematical thinking, and cultivate afresh his mathematical imagination, as a prelude to cultivating more effectively the mathematical imagination of his pupils (see page 170). It is acknowledged that when observing children it is easier to describe the very slow learner on the one hand, and, on the other, the initiative and powers of expression of the markedly able, than it is to give any picture of the children between: on the whole, too, more opportunities have occurred for recording the individual experiences of the under-sixes. The chapter (VIII) on Written Reckoning has but few references to actual incidents, but will repay careful reading and provoke thought: it contains an analysis of necessary mental preparation for reckoning.

H. M. C.

A Statistical Primer. By F. N. DAVID Pp. x, 226. 22s. 1953. (Charles Griffin)

Dr. David was the author of *Probability Theory of Statistical Methods*, reviewed in M.G. xxxiv (308) 155 (1950, May). During recent years she has lectured to a mixed audience, of postgraduates and undergraduates drawn from nearly all science departments, at University College, London; the present book has been composed from her lectures, and is dedicated to E.M.E., who will be recognised as one of the earliest and most helpful of the colleagues of Karl Pearson. " It is hoped that anyone wishing to learn the elements of statistics will find the book useful, from the research worker to the teacher of the secondary school." The book is professedly for the non-mathematician, and the mathematics is slight, but I think that the teacher in the secondary school, unless he is in a grammar school, will find that the ideas are too difficult for his pupils.

The book falls into two main parts. Chapters 1 to 8 deal with descriptive statistics. These deal with qualitative and quantitative aggregates, and the measures appropriate for considering their location, dispersion, skewness, and curtosis. The other chapters, from Ch. 9 to 24, work on the notion of probability and of sampling. There are chapters on the various tests of significance—the t test, the χ^2 test and the F test: these take us to Ch. 17. After that this second part finishes with the binomial, and its approximations by the normal and Poisson, and with tests associated with it. The writer is particu-

larly good when considering the conditions to be satisfied in the various cases, when normality may be assumed, when one-tailed and when two-tailed probabilities are appropriate, and freely advises when the test is inconclusive and further data should be obtained. Chapter 23 is about the Index of Dispersion as a test for material with " contagion " in the data, which I do not remember being dealt with in any comparable book. The book concludes with an index.

There is no bibliography, though a few books are referred to in the text, usually with, perhaps, insufficient reference (no publisher, no date, etc.), whilst I am not able to find that Molina's table is in the volume mentioned on p. 199 : I think that it is published by an American house. The tables of examples and the tables of reference are numbered in one series, so that the latter have to be sought out in the text. They are Table 10.4—normal curve ordinates—, 11.4—normal curve areas—, 14.1, for t, 15.1, for χ^2—the last two being based on fuller *Biometrika* tables still to be published—, 17.1, for F, and 18.2—Pascal's triangle : they are summary tables for the most part and are not always sufficient for working out some of the examples. The same comment, about ease of reference, applies to the numbering of the diagrams of statistical functions in the same series as those accompanying the examples. There are no exercises for the student : the one mentioned in the text as appropriate for the reader to work out (p. 221) did not give, when this reader tried it, the anticipated result suggested by the author. It seems a pity that the " judgment by eye " as, e.g., on p. 203, is not checked by the simple graphical method given by the use of the Poisson Probability paper that is obtainable from certain stationers.

The book is nicely got up, and misprints are few and not likely to be misleading : the commonest is a failure to insert the final bracket in rather lengthy expressions.

I think that it is fair to sum the book up as a sound elementary treatise, of which the first part certainly is suitable for any non-specialist sixth form class interested in the topic. The examples are such as should be of general interest to any keen pupil : they are mostly derived from elementary biology, at almost the " natural history " level. The whole volume should certainly be considered in connection with school statistical courses.

FRANK SANDON.

Sir Isaac Newton. By E. N. DA C. ANDRADE. Pp. 140. 7s. 6d. 1954. Brief Lives, 11 (Collins)

In this brilliant addition to an admirable series, Andrade presents a sparkling miniature. Without symbols, in simple language which a bright 14-year-old could understand, he depicts pre-Newtonian science and then the tremendous impact of the giant on the doorways of natural philosophy, the great discoveries in mathematics, gravitation and optics. Moreover, he properly emphasises the almost single-handed effort which swung the world of science away from asking questions such as " What is light? " to asking and answering the question " How does light behave? " Essential items of personal biography are given their due and no more ; the paradoxes of mind and character are not concealed. " High minded and petty ; modest and overbearing ; " and the final judgement : " the first modern scientist and the last of the mages."

L. T. More's standard life is not now readily obtainable ; but even where it is accessible, Andrade's volume is not superfluous, since it provides the best short and simple account, particularly for the general reader, that I have seen. Every school library should have a copy. T. A. A. B.

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